1. Formulation

- linear program (LP)
- integer program (IP)
- non-linear program (NLP)

→ LP

- def. linear function

$f: \mathbb{R}^n \to \mathbb{R}$. a function of the form $f(\vec{x}) = a_1x_1 + \cdots + a_nx_n$

- def. affine function.

$f: \mathbb{R}^n \to \mathbb{R}$ a linear function + constant

ep. affine func in 2D : $f(x,y) = Ax + By + C$
affine func in 3D : $f(x,y) = Ax + By + Cz + D$

- def. linear program

max/minimize an affine function subject to a finite number of linear constraints.

$f(x) \le \beta$ $f(x) \ge \beta$ $f(x) = \beta$ ($f(x)$ → linear func. $\beta \in \mathbb{R}$)

ep. min $x_1 + \frac{1}{x_2}$ → not a function ← $\cos(x)$ $\boxed{X}$

s.t. $x_1 + x_2 < 3$ → 符号

$x_1 + \alpha x_2 \le 5$ $\forall \alpha \in \mathbb{R}$ → too many constraints

→ IP

每个 constraints 是整数

Example 1.3.1. ***KitchTech Shipping***

A company wishes to ship crates from Toronto to Kitchener. Each crate type has a weight and a value:

Type	1	2	3	4	5	6
weight (lbs)	30	20	30	90	30	70
value ($)	60	70	40	70	20	90

The total weight of crates shipped must not exceed 10,000 lbs. The goal is to ***maximizes*** *the total value of shipped goods.*

variables : x_i denotes the number of crate type i

objective function : $\max\ 60x_1+70x_2+40x_3+70x_4+20x_5+90x_6$

constraints : $30x_1+20x_2+30x_3+90x_4+30x_5+70x_6 \le 10000$

$x_i \geq 0 \quad \forall i \in [1,6]$

$x_i \in \mathbb{Z} \quad \forall i \in [1,6]$

Example 1.3.2. ***KitchTech: Additional Conditions***

Suppose that we must not send more than 10 crates of the same type, and we can only send crates of type 3, if we send at least 1 crate of type 4. Note that we can send at least 10 crates of type 3 by the previous constaints!

new constraints: $x_i \le 10 \quad \forall i \in [1,6]$

$x_3 \le 10x_4$

Example 1.3.3. ***KitchTech: 1 more tricky case***

Suppose that we must

- *take a total of at least 4 crates of type 1 or 2, or*
- *take at least 4 crates of type 5 or 6*

new variable : y.

$$\begin{cases} y=1 \Rightarrow x_1+x_2 \geq 4 \\ y=0 \Rightarrow x_5+x_6 \geq 4 \end{cases}$$

→ binary variable

用于 logical constraints

eg. $\overset{y}{A}$ or $\overset{1-y}{B}$. → C

A and B → C .

new constraints : $x_1+x_2 \geq 4y$

$x_5+x_6 \geq 4\underline{(1-y)}$

$0 \le y \le 1$

$y \in \mathbb{Z}$

→ Graph (属于 IP)

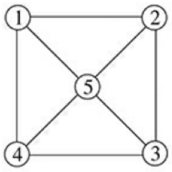
$G = (V, E)$

↙ vertices ↘ edges

vertices : u, w ⋯ ∈ V

edges : uw, wz, ⋯ ∈ E.

ep.



$V = \{1, 2, 3, 4, 5\}$.

$E = \{12, 23, 34, 15, 25, 35, 45, 14\}$

2 vertices u & v are adjacent if uv ∈ E.

u & v are endpoint of edge uv ∈ E.

edge e ∈ E is incident to u ∈ V

- s-t path

s-t path is a sequence of edges v_1v_2, v_2v_3, ⋯, $v_{k-1}v_k$

s.t. 1. 从 s 开始, 到 t 结束 $v_1 = s$, $v_k = t$

2. 不重复经过点

* empty sequence is a s-s graph

length of a path $P = v_1v_2, \cdots, v_{k-1}v_k$ is the sum of length of edges on P.

$c(P) = \sum(c_e : e \in P)$

- Shortest path problem

题目： Given $G=(V,E)$. length $c_e \geq 0$ $\forall e \in E$. $s,t \in V$.

Find min length / s, t-path P

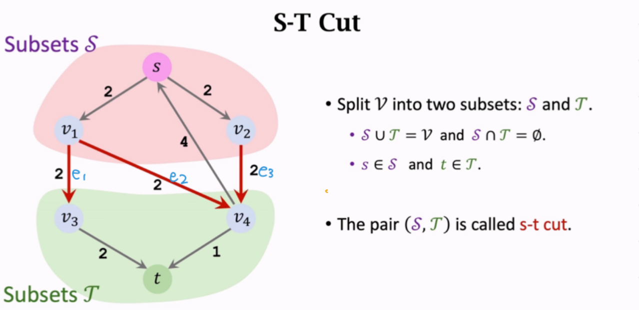
方法： s, t- cut (source-to-tangent cut)

Given a graph $G=(V,E)$ and 2 vertices $s,t \in V$. $s \neq t$

then for $S \subseteq V$, the set $\delta(S) := \{ w, v \in E : \underline{w \in S \quad v \notin S} \}$

is called s-t cut

S-T Cut



S-T Cut

- Split $\mathcal{V}$ into two subsets: $\mathcal{S}$ and $\mathcal{T}$.
  - $\mathcal{S} \cup \mathcal{T} = \mathcal{V}$ and $\mathcal{S} \cap \mathcal{T} = \emptyset$.
  - $s \in \mathcal{S}$ and $t \in \mathcal{T}$.
- The pair $(\mathcal{S}, \mathcal{T})$ is called s-t cut.

1. 先分成包含 s 和包含 t 的两个版块

$S \cup T = V$, $S \cap T = \emptyset$

$s \in S$, $t \in T$

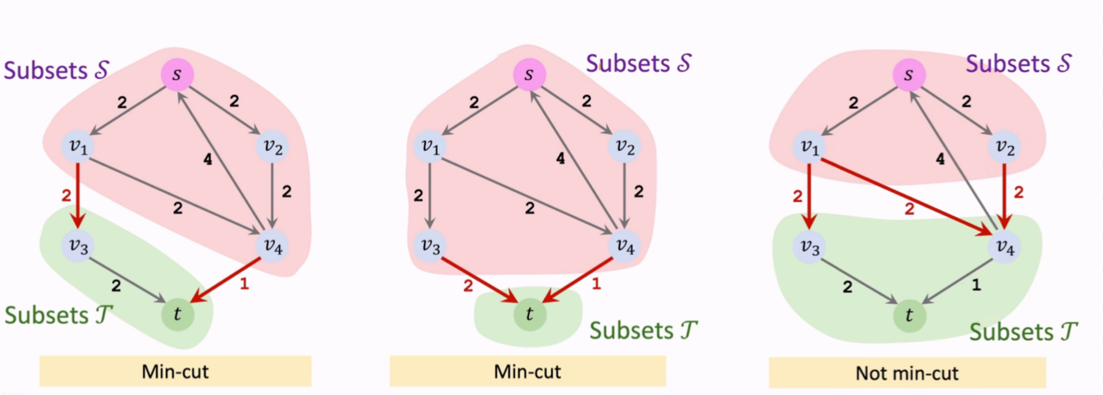
2. 看需要切开的，分离 S 和 T 的 edge

Example

如图 $\delta(\{t, v_3, v_4\}) = \{e_1, e_2, e_3\}$

是很多种的一种 s-t cut

Min-Cut



$\underbrace{\qquad\qquad\qquad\qquad}_{\text{min-cut 不唯一}}$

不是 min-cut

* If P is s.t-path. $\delta(S)$ is s.t cut, then P must have an edge from $\delta(S)$

* If $S \subseteq E$ contains at least one edge from every s,t-cut, then S contains s,t-path

☆

binary variable

variables : x_e for each $e \in E$. denote $\underline{x_e} = \begin{cases} 1 & e \in P \\ 0 & e \notin P. \end{cases}$

objective : min $\sum(c_e x_e : e \in E)$

constraints : 对于每个 s,t-cut $\delta(U)$，都有一个 constraint $\underline{\sum(x_e : e \in \delta(U)) \geq 1}$

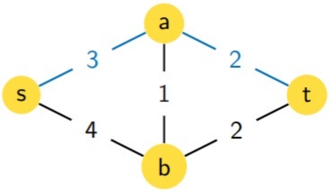
P have an edge from $\delta(U)$

$\sum(x_e : e \in \delta(U)) \geq 1$. $U \subseteq V$. $s \in U$, $t \notin U$

$x_e \geq 0$ $x_e \in \mathbb{Z}$

$\Downarrow$

We have the graph to the right, and want to find a perfect matching with minimum cost.



$$\begin{array}{ccccccc} & sa & sb & ab & at & bt & \\ \text{objective} : & (3, & 4, & 1, & 2, & 2) & x \end{array}$$

constraints :
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_{sa} \\ x_{sb} \\ x_{ab} \\ x_{at} \\ x_{bt} \end{pmatrix} \geq 1$$

所有可能的路径

$x \geq 0$ $x \in \mathbb{Z}$

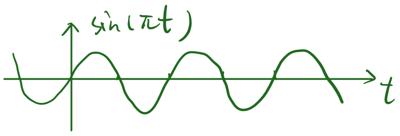
→ NLP

A **nonlinear program (NLP)** is of the form

$$
\begin{array}{ll}
\min & f(x) \\
\text{s.t.} & g_1(x) \le 0 \\
& g_2(x) \le 0 \\
& \cdots \\
& g_m(x) \le 0
\end{array}
$$

where

- $x \in \mathbb{R}^n$
- $f : \mathbb{R}^n \to \mathbb{R}$
- $g_i : \mathbb{R}^n \to \mathbb{R}$



eg. $\min \quad x_1 + x_2 - x_3$

subject to

$$\sin(\pi x_1) = 0$$
$$\sin(\pi x_2) = 0 \quad \longrightarrow \quad x_1, x_2 \in \mathbb{Z}$$
$$x_3(1 - x_3) = 0 \quad \longrightarrow \quad x_3 \in \{0, 1\}$$
$$x_1 + 2x_2 + 3x_3 \le 0$$

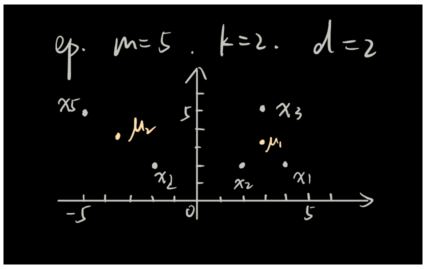
* every $\underset{(包括\ LP)}{\underline{\mathbb{P}}}$ can be modelled as NLP.

– Fermat's last theorem

不存在 integer $x, y, z \geq 1$ $\quad n \geq 3$. s.t

$$x^n + y^n = z^n$$

Q. Given points $x_1, \cdots, x_n \in \mathbb{R}^d$, partitions these points into k ($k \le m$). groups $S_1, S_2, \cdots S_k$. s.t. $\sum_{i=1}^{k} \sum_{y \in S_i} \|y - \mu_i\|^2$ is minimized. (μ_i. $i=1, \cdots, k$) is the mean of $\{y \in \mathbb{R}^d. y \in S\}$.



$$\mu_1 = \frac{1}{3}\binom{2}{2} + \frac{1}{3}\binom{4}{2} + \frac{1}{3}\binom{3}{5} = \binom{3}{3}$$

$$\mu_2 = \frac{1}{2}\binom{-3}{2} + \frac{1}{2}\binom{-5}{4} = \binom{-4}{3}$$

$$\sum_{\substack{y \in S_1 \\ = \{x_1, x_2, x_3\}}} \|y - \mu_1\|^2 = (\sqrt{2})^2 + (\sqrt{2})^2 + 2^2 = 8$$

$$\sum_{\substack{y \in S_2 \\ = \{x_4, x_5\}}} \|y - \mu_2\|^2 = (\sqrt{2})^2 + (\sqrt{2})^2 = 4.$$

$\downarrow$

total 12.

variables:

let $\mu_i \in \mathbb{R}^d$. $i=1, \cdots, k$. represent the mean for S_i

let z_{ij}. $i=1, \cdots, k$. $j=1, \cdots, m$. indicate whether x_j is in S_i

objective $\sum_{i=k}^{k} \sum_{j=1}^{m} z_{ij} \|x_j - \mu_i\|^2$

constraints. $\sum_{i=1}^{k} z_{ij} = 1 \quad \forall j = 1, \cdots, n.$

$z_{ij}(1 - z_{ij}) = 0 \quad \forall i = 1, \cdots, k. \quad j = 1, \cdots, n.$

$(\sum_{j=1}^{m} z_{ij}) \mu_i = \sum_{j=1}^{m} z_{ij} x_j \quad \forall i = 1, \cdots, k.$

2. 解 LP.

proof

- optimal solution : 最优解.
- feasible solution : 满足所有条件 in 解
- unbounded $\forall M$, $\exists$ an feasible sol with objective value greater / smaller than M.

- Ferkas' Lemma.

There is no solution to $Ax = b$, $x \geq 0$.

if $y^T\underline{A} \geq 0^T$ $y^T\underline{b} < 0$

- Proposition : unbounded

Let A be matrix. b & c be vectors.

Then the LP $\boxed{\begin{array}{ll} \max & c^Tx \\ \text{subject to} & Ax=b \\ & x \geq 0 \end{array}}$ is unbounded

if there exist a $\underline{\text{feasible } \bar{x} \text{ and vector } d}$. s.t.
(≥ 0) ↑

1) $Ad = 0$
2) $d \geq 0$
3) $c^Td > 0$

certificate of unboundediess for the LP

proof:

→ consider the sequence of vectors $\tilde{x}(t) = \bar{x} + t \cdot d$. indexed by $t \in \mathbb{R}$. $t \geq 0$.

for every $t \geq 0$. $\tilde{x}(t)$ is feasible indeed. $\tilde{x}(t) = \underbrace{\bar{x}}_{\geq 0 \text{ by feasibility of } \bar{x}} + \underbrace{t\,d}_{\geq 0 \text{ (by 2))}} \geq 0$.

→ Also,

$$
\begin{aligned}
& A\,\tilde{x}(t) \\
&= A(\bar{x} + t \cdot d) \\
&= A\,\bar{x} + A(t \cdot d) \\
&= \underbrace{A\bar{x}}_{= b \text{ by feasibility of } \bar{x}} + t \cdot \underbrace{(A\,d)}_{= 0 \text{ by 1)}} \\
&= b
\end{aligned}
$$

← 证明是 solution

$$
\begin{aligned}
& \lim_{t \to +\infty} \text{value}(\tilde{x}(t)) \\
&= \lim_{t \to \infty} c^T(\tilde{x} + t\,d) \\
&= \lim_{t \to \infty} \underbrace{c^T \bar{x}}_{\text{independ on } t} + \underbrace{(c^T d)}_{> 0 \text{ by 3)}}\, t \\
&= \infty
\end{aligned}
$$

← 证明 unbounded

$\therefore$ LP is unbounded.

Q.

$$
\boxed{
\begin{aligned}
&\max \quad z(x) := (-1 \quad -4 \quad 0 \quad 0)\,x + 4 \\
&\text{subject to} \quad \begin{pmatrix} -1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\
&\qquad\qquad\qquad x \geq 0
\end{aligned}
}
$$

Prove $\bar{x} = (0 \quad 0 \quad 4 \quad 5)$ is feasible sol and 4 is upper bound

consider x' be an arbitrary feasible solution,

$$z(x') = \underbrace{(-1 \quad -4 \quad 0 \quad 0)}_{\leq 0}\, \underbrace{x'}_{\geq 0} + 4 \leq 4$$

Q. Prove

$$\boxed{\begin{aligned} &\max\ (3, 4, -1, 2)^T x \\ &\text{s.t. } \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \\ &x \geq 0 \end{aligned}}$$

is infeasible

Show no solution!

$$\begin{cases} [-3\ \ 2\ \ 6\ \ 7]\, x = 6 & ① \\ [\ 4\ \ -2\ \ -4\ \ 8]\, x = 4 & ② \end{cases}$$

$\underbrace{-1 \times ① + 2 \times ②}$ 得 $[1\ \ 0\ \ 2\ \ 1]\, x = -2$

Suppose $\exists\, \bar{x} \geq 0$ satisfying ① & ②, Then $\bar{x}$ satisfy

$\because\ \underset{\geq 0}{\underset{\downarrow}{\underline{[1\ \ 0\ \ 2\ \ 1]}}}\ \underset{\geq 0}{\underset{\downarrow}{x}} = \underset{<0}{\underset{\downarrow}{-2}}$ contradicts.

使得一边 ≥ 0 一边 < 0.

Q. Prove

$$\boxed{\begin{aligned} &\max\ (2\ \ 0\ \ 1\ \ -1)^T x \Leftarrow z \\ &\text{subject to } \begin{pmatrix} 1 & 1 & -1 & 1 \\ -2 & 0 & 3 & -4 \\ 3 & 2 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 3 \\ -4 \\ 8 \end{pmatrix} \\ &x \geq 0 \end{aligned}}$$

is unbounded.

记 $t \to \infty$, $z \to \pm\infty$

consider a feasible solution $\bar{x} = (2\ \ 1\ \ 0\ \ 0)^T$.

→ consider a vector $d = (1\ 0\ 2\ 1)^T$ and a sequence of vectors $\tilde{x}(t) = \bar{x} + t \cdot d$ indexed by $t \in \mathbb{R}$. $t \geq 0$. i.e. $\tilde{x}(t) = (2+t\ \ \ 1+0 \cdot t\ \ \ 0+2t\ \ \ 0+1 \cdot t)^T$.

→ for every $t \geq 0$. the vector $\tilde{x}(t)$ is feasible.

$2+t \geq 0$. $1 \geq 0$. $2t \geq 0$. $t \geq 0$. whenever $t \geq 0$

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ -2 & 0 & 3 & -4 \\ 3 & 2 & 0 & -3 \end{pmatrix} \begin{pmatrix} 2+t \\ 1 \\ 2t \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 8 \end{pmatrix}$$

$$\rightarrow \lim_{t \to +\infty} (2\ \ 0\ \ 1\ \ -1)^T (\tilde{x}(t)) = 2(2+t) + 0 \cdot 1 + 1 \cdot 2t + (-1) \cdot t$$

$$= \lim_{t \to +\infty} (4+3t) = +\infty$$

$\therefore$ the LP is unbounded.

→ Standard equality form (SEF)

- def. SEF

$$\boxed{\begin{array}{ll} \underline{\underline{\max}} & c^T x + \bar{z} \\ \text{s.t} & Ax \underline{\underline{=}} b \\ & \underline{\underline{x}} \geq \mathbb{0} \end{array}}$$

← "max" problem

← all constraints except $x_j \geq 0$ is equality constraints

← $x_j \geq 0 \quad \forall j.$

- def. equivalent program.

① (P) is infeasible ⟺ (P') is infeasible

② (P) is unbounded ⟺ (P') is unbounded.

③ any optimal solution of (P) can be efficiently convert into an optimal sol of (P') and vice versa

Q. convert the following LP in an <u>equivalent</u> SEF. (P')

$$\min \begin{pmatrix} 1 & 2 & -3 \end{pmatrix} x$$

$$\text{sub. to} \quad \begin{pmatrix} 1 & -2 & 7 \\ 2 & 0 & 5 \\ 3 & 2 & 3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \qquad (P)$$

$$x_1 \leq 0 \qquad x_2 \geq 0. \qquad x_3 \text{ free}$$

→ Change min ⇒ max

$$\max \begin{pmatrix} -1 & -2 & 3 \end{pmatrix} x.$$

→ Change inequalities ⇒ equalities $\geq / \leq \rightarrow =$

$1x_1 - 2x_2 + 7x_3 \geq 2.$ equivalent to $1x_1 - 2x_2 + 7x_3 - x_4 = 2.$ for some $x_4 \geq 0$

$2x_1 + 0x_2 + 5x_3 \leq 4$ equivalent to $2x_1 + 5x_3 + x_5 = 4$ for some $x_5 \geq 0$

slack variable

$$\max \begin{pmatrix} -1 & -2 & 3 & 0 & 0 \end{pmatrix} x.$$

$$\text{sub. to} \quad \begin{pmatrix} 1 & -2 & 7 & -1 & 0 \\ 2 & 0 & 5 & 0 & 1 \\ 3 & 2 & 3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

$$x_1 \leq 0 \qquad x_2 \geq 0 \qquad x_3 \text{ free} \qquad x_4 \geq 0. \qquad x_5 \geq 0$$

→ Change nonpositive variable ⇒ nonnegative $- \rightarrow +$

replace x_1 by a new variable $\tilde{x}_1$. s.t. $x_1 = -\tilde{x}_1$.

after the change, we can remove $\tilde{x}_1$.

$$\max \begin{pmatrix} 1 & -2 & 3 & 0 & 0 \end{pmatrix} x.$$

$$\text{sub. to} \quad \begin{pmatrix} -1 & -2 & 7 & -1 & 0 \\ -2 & 0 & 5 & 0 & 1 \\ -3 & 2 & 3 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

$$x_1 \geq 0 \qquad x_2 \geq 0 \qquad x_3 \text{ free} \qquad x_4 \geq 0. \qquad x_5 \geq 0$$

→ Change free variables → 2个 nonnegative variable 的差.

x_3 free. is equivalent to $3=3-0$
$-2=0-2$
$-2=3-5$

$x_3 = x_3^+ - x_3^-$ for some $x_3^+ \geqslant 0$. $x_3^- \geqslant 0$

$$\max \begin{pmatrix} 1 & -2 & 3 & -3 & 0 & 0 \end{pmatrix} x.$$

$$\text{sub. to} \begin{pmatrix} -1 & -2 & 7 & -7 & -1 & 0 \\ -2 & 0 & 5 & -5 & 0 & 1 \\ -3 & 2 & 3 & -3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

$x \geqslant 0$

→ Basis

- def. basis (B)

B: subset of column indices.

A_B: column sub-matrix of A indexed by set B.

A_j : col j of A.

① A_B is square matrix

② A_B is non-singular (colums independent)

ep. $A = \left(\begin{array}{c|c|c|c|c|c} 2 & 3 & 0 & -1 & 6 & 4 \\ -1 & 0 & 5 & 12 & 20 & -2 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}\right)$

- $B = \{1, 4\}$ is a basis $A_B = \begin{pmatrix} 2 & -1 \\ -1 & 12 \end{pmatrix}$ is square nonsingular
- $B = \{1, 6\}$ is not a basis $A_B = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$ is not nonsingular.
- $B = \{1, 2, 3\}$ is not a basis A_B is not square

- def. basic solution

① $Ax = b$

② $x_j = 0 \quad \forall j \notin B$.

vector $\vec{x}$ is a basic solution of $Ax = b$

$$\begin{tikzcd}\text{Bases} \arrow[r, shift left, "\text{① Given basis B. there is unique basic sol} \quad \bar{x}_N = 0"] & \text{basic sol} \arrow[l, shift left, "\text{② Given a basic sol. there may have lot of basis}"]\end{tikzcd}$$

① proof: $b = Ax = \sum_j A_j x_j = \sum_{j\in B} A_j x_j + \sum_{j\notin B} A_j \underline{\underline{x_j}}$ $=0$

$$= \sum_{j\in B} A_j x_j$$

$$= A_B x_B.$$

$\because$ B is basis $\therefore A_B$ is non-singular. A_B^{-1} exists.

$\therefore x_B = A_B^{-1} b.$

$\bar{x}_N = 0$

$$b = A\bar{x} = A_B \cdot \bar{x}_B + A_N \underbrace{\bar{x}_N}_{=0} = \underline{A_B}\bar{x}_B$$

square non-singular

$\bar{x}_B = A_B^{-1} b$

$$\boxed{\begin{array}{c}\bar{x}_B \\ \hline \bar{x}_N\end{array}} = \bar{x}.$$

basic variable

not basic

② consider in SEF: $\max\{c^T x : Ax = b.\ x \geq 0\}.$

If rows are dependent, then either.
- no solution
- 其中一个 constraint 多余

* A solution $\bar{x}$ for $A\bar{x} = \bar{b}$ is a basic if $\{i : \bar{x}_i \neq 0\}$ corresponds to linearly independent columns of A

Q. Find basic solution for $\underbrace{\begin{pmatrix}1&0&1&-1\\0&1&1&1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}2\\2\end{pmatrix}}_{b}$ when $B=\{1,4\}$?

$$\rightarrow \begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}1&0&1&-1\\0&1&1&1\end{pmatrix} x$$

$$= x_1\begin{pmatrix}1\\0\end{pmatrix} + \overset{=0}{\underline{\underline{x_2}}}\begin{pmatrix}1\\0\end{pmatrix} + \overset{=0}{\underline{\underline{x_3}}}\begin{pmatrix}1\\0\end{pmatrix} + x_4\begin{pmatrix}-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}x_1\\x_4\end{pmatrix}$$

$$\rightarrow \begin{pmatrix}x_1\\x_4\end{pmatrix} = \begin{pmatrix}1&-1\\0&1\end{pmatrix}^{-1}\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$$

$\rightarrow$ basic solution : $(4 \quad 0 \quad 0 \quad 2)^T$

ep. take $A = \begin{pmatrix}2&3&0&-1&6&4\\-1&0&5&12&20&-2\end{pmatrix}$. $b=(3 \quad 10)^T$

- $\bar{x} = (2, \underline{0, 0}, 1, \underline{0, 0})^T$ is a basic solution for $B=\{1,4\}$.

  $x_N = \mathbb{0} = (0 \quad 0 \quad 0 \quad 0)^T$. $N=\{2,3,5,6\}$. $\leftarrow$ (# 列)

- $\bar{x} = (0 \quad 1 \quad 2 \quad 0 \quad 0 \quad 0)^T$ is a basic solution for $B=\{2,3\}$

  $N=\{1,4,5,6\}$.

- $\bar{x} = (0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{2} \quad 0)^T$ is a basic solution for $B=\{5,6\}$

  $N=\{1, 2, 3, 4\}$

- $\bar{x} = (-2 \quad 0 \quad 0 \quad -1 \quad 1 \quad 0)^T$ is a solution but not a basic solution

→ Canonical form

- def. canonical form.

we say an LP
$$\boxed{\begin{array}{rl} & \max\ c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}}$$

is canonical form for a basis of A. if 1) $c_j = 0 \quad \forall j \in B$

2) A_B is $I_{n \times n}$

eg.

$$\begin{pmatrix} 5 & 1 & 4 & 4 & \boxed{0 \quad 0 \quad 0} \end{pmatrix} \leftarrow c_j = 0$$

$$\begin{pmatrix} 4 & 1 & 3 & 5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 3 & 0 & 0 & 1 \end{pmatrix}$$

$\uparrow A_B = I_{n \times n}$

Procedure 1: Find Canonical Form

Consider A with basis B,

(P)

$$\begin{array}{rl} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq \mathbb{0} \end{array}$$

(P')

$$\begin{array}{rl} \max & \underbrace{[c^T - y^T A]}_{\bar{c}^T} x + \underbrace{y^T b}_{\bar{z}} \\ \text{s.t.} & \underbrace{A_B^{-1} A}_{A'} x = \underbrace{A_B^{-1} b}_{b'} \\ & x \geq \mathbb{0} \end{array}$$

where $y = A_B^{-T} c_B$, then

1. (P') is in canonical form for basis B, i.e. $\bar{c}_B = 0$ and $A'_B = I$
2. (P) and (P') have the same feasible region
3. **feasible solutions** have the same objective value for (P) and (P').

- For all basis B. ∃ P' in canonical form of B s.t.
  1. P & P' have same feasible region
  2. feasible sols have the same objective value for P & P'.

Q. We have the LP model:

$$\boxed{\begin{array}{l} \max \ \overset{c}{\underline{(0 \ \ 0 \ \ 2 \ \ 4)}}\,x \\ \text{s.t} \ \ \underbrace{\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{b} \end{array}}$$

Rewrite (P) in canonical form for basis $B = \{2, 3\}$

→ Replace $Ax = b$ by $A'x = b'$ with $A'_B = I$

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & \underset{\uparrow}{1} & \underset{\uparrow}{1} & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad\qquad Ax = b$$

$$\underline{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1}} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \underline{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad\qquad \underbrace{A_B^{-1} A}_{A'} x = \underbrace{A_B^{-1} b}_{b'}.$$

$$\begin{pmatrix} -1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

→ Replace $c^T x$ by $\bar{c}^T x + \bar{z}$. with $\bar{c}_B = 0$.

- $0 = -(y_1, y_2) \overset{Ax}{\underline{\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x}} + (y_1, y_2) \overset{b}{\underline{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}} \qquad z = [\underbrace{c^T - y^T A}_{\bar{c}^T}] x + \underbrace{y^T b}_{\bar{z}}$

  $z = (0 \ \ 0 \ \ 2 \ \ 4) x.$

  $= [\underbrace{(0 \ \ 0 \ \ 2 \ 4) - (y_1 \ \ y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix}}_{\bar{c}^T}] x + \underbrace{(y_1 \ \ y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\bar{z}}$

- $(0 \ \ 0) = \bar{c}_B^T = (0 \ \ 2) - (y_1 \ \ y_2) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad 0^T = -\bar{c}_B^T = c_B^T - y^T A_B$

  $(y_1 \ \ y_2) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = (0 \ \ 2) \qquad y^T A_B = c^T{}_B$

  $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \qquad A_B^T y = c_B$

  $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad y = (A_B^T)^{-1} c_B = A_B^{-T} c_B.$

- choose $(y_1, y_2) = (2, 0) \qquad \therefore z = (-2 \ \ 0 \ \ 0 \ \ 6) x + 2$

$$\boxed{\begin{array}{l} \max \ (-2 \ \ 0 \ \ 0 \ \ 6) x + 2 \\ \text{s.t} \ \begin{pmatrix} -1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \qquad\quad x \geq 0 \end{array}}$$

Q. Find equivalent canonical form with respect to $B=\{3,4\}$

for

$$\boxed{\begin{aligned} &\max \ \underbrace{\begin{pmatrix} 2 & -3 & -2 & 1 \end{pmatrix}}_{c} x \\ &\text{subject to} \\ &\qquad \underbrace{\begin{pmatrix} -4 & 1 & 3 & -1 \\ 3 & 0 & -2 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 7 \\ -4 \end{pmatrix}}_{b} \\ &\qquad x \geq 0 \end{aligned}}$$

$$A_B = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \qquad A_B^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -4 & 1 & 3 & -1 \\ 3 & 0 & -2 & 1 \end{pmatrix} x = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

for every feasible x, we have.

$$\begin{aligned} (2 \ \ -3 \ \ -2 \ \ 1)x &= (2 \ \ -3 \ \ -2 \ \ 1)x \\ &\quad + \underset{\sim}{2}(-1 \ \ 1 \ \ 1 \ \ 0)x - 2\cdot 3 \\ &\quad - \underset{\sim\sim\sim}{\quad}(1 \ \ 2 \ \ 0 \ \ 1)x + 2 \\ &= (-1 \ \ -3 \ \ \boxed{0 \ \ 0})x - 4 \end{aligned}$$

(目的：使后面 B 变0)

$$\boxed{\begin{aligned} &\max \ (-1 \ \ -3 \ \ 0 \ \ 0)x - 4 \\ &\text{subject to} \ \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &\qquad\qquad x \geq 0 \end{aligned}}$$

→ Simplex method.

- def. • a basic solution $\bar{x}$ is <u>feasible</u> if $\bar{x} \geq 0$
  - • a bisis B is called <u>feasible</u> if the corresponding basic sol. $\bar{x}$ is feasible ep. $A_B^{-1} b \geq 0$

↓↓ Optimal solution

Q.

$$\max \; \left(5 \quad 1 \quad 4 \quad 4 \quad \overset{C_B}{\boxed{0 \quad 0 \quad 0}}\right) x$$

canonical form for $B = \{5, 6, 7\}$

subject to

$$\begin{array}{c} \\ \\ \\ B = \end{array} \left(\begin{array}{cccc|ccc} 4 & 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 3 & 0 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \right) x = \begin{pmatrix} 17 \\ 4 \\ 10 \end{pmatrix}$$

A_B = columns 5, 6, 7 (boxed)

$$x \geq 0$$

→ The current feasible basis $B = \{5, 6, 7\}$. with sol $(0, 0, 0, 0, 17, 4, 10)^T$

$x_1 . x_2 . x_3 . x_4$ non basic

$x_5 . x_6 . x_7$ basic

bound

$$z = \underset{\pi}{5} x_1 + 1 \cdot x_2 + 4x_3 + 4x_4.$$

$$\tfrac{17}{4} \rightarrow x_5 = \boxed{17} - \boxed{4} x_1 - x_2 - 3x_3 - 3x_4$$

$$4 \rightarrow x_6 = 4 - 1 x_1 - 1 x_3 - 1 \cdot x_4$$

$$5 \rightarrow x_7 = 10 - 2x_1 - x_2 - 2x_3 - 3x_4$$

↓ min is 4

目的：去掉 x_6. entre x_1

→ The current feasible basis $B = \{1, 5, 7\}$. with sol $(4, 0, 0, 0, 1, 0, 2)^T$

$x_2 . x_3 . x_4 . x_6$ nonbasic

$x_1 \quad x_5 . x_7$ basic

$$z = 20 + 1 \cdot x_2 - 1 \cdot x_3 - 1 \cdot x_4 - 5x_6$$

$$x_6 \quad 1 \rightarrow x_1 = 4 - 1 \cdot x_3 - 1 \cdot x_4 - 1 \cdot x_6$$

$$\emptyset \rightarrow x_5 = 1 - 1 \cdot x_2 + 1 \cdot x_3 - 1 \cdot x_4 + 4 \cdot x_6$$

$x_5 \div \frac{17}{4} - x_6 \div 4$

$$2 \rightarrow x_7 = 2 - 1 \cdot x_2 - 1 \cdot x_4 + 2 \cdot x_6$$

$x_7 \div 5 - x_6 \div 4$

↓ min is 1

目的：去掉 x_5. entre x_2.

for x_7 $\quad x_7 = 10 - 2x_1 - 1 \cdot x_2 - 2x_3 - 3x_4 = 2 - x_2 - x_4 + 2x_6$ (目的：消 x_1)

→ The current feasible basis $B = \{1, 2, 7\}$ with sol $(4, 1, 0, 0, 0, 0, 1)$

$$z = \boxed{21} \qquad\qquad - 1 \cdot x_5 - 1 \cdot x_6$$

$$x_1 = 4 - 1 \cdot x_3 - 1 \cdot x_4 \qquad - 1 \cdot x_6$$

$$x_2 = 1 + 1 \cdot x_3 + 1 \cdot x_4 - 1 \cdot x_5 + 4 \cdot x_6$$

$$x_7 = 1 - 1\, x_3 - 2 x_4 + 1 \cdot x_5 - 2 x_6$$

← no positive coefficient for variables

STOP

∴ The value is 21

So $\bar{x} = (4, 1, 0, 0, 0, 0, 1)^T$ is optimal with value 21.

$B = \{1, 2, 7\}$ is optimal basis

The certificate of optimality is $y = \boxed{A_B^{-T} c_B} = \left(\begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}^T \right)^{-1} \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

↑ with respect to the original LP

If original LP is canonical form, there is an easier way to compute certificate of optimality

Here, take $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ multiply by $\boxed{-1}$ ← always −1

iii unbounded

Q. Solve $\max \begin{pmatrix} 1 & 3 & -1 & 0 & 0 & 0 \end{pmatrix} x$

subject to $\begin{pmatrix} 2 & 2 & -1 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 1 & -3 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$

$x \geqslant \mathbb{0}$

→ Current feasible basis $B = \{4, 5, 6\}$ with sol $(0, 0, 0, 10, 10, 10)^T$

$z = 1x_1 + 3x_2 - 1x_3$

5 → $x_4 = 10 - 2x_1 - 2x_2 + 1 \cdot x_3$

$\frac{10}{3}$ → $x_5 = 10 - 3x_1 + 2x_2 - 1 \cdot x_3$

10 → $x_6 = 10 - 1x_1 + 3x_2 - 1 \cdot x_3$

min $\frac{10}{3}$

x_1 entre. x_5 leave.

→ Current feasible basis $B = \{1, 4, 6\}$. with sol $(\frac{10}{3}, 0, 0, \frac{10}{3}, 0, \frac{20}{3})^T$

$z = \frac{10}{3} + \frac{11}{3} x_2 - \frac{4}{3} x_3 - \frac{1}{3} x_5$

$x_2\uparrow$ $x_1\uparrow$!!! $\not{0}$ → $x_1 = \frac{10}{3} \underset{\triangle}{+} \frac{2}{3} x_2 - \frac{1}{3} x_3 - \frac{1}{3} x_5$.

1 → $x_4 = \frac{10}{3} - \frac{10}{3} x_2 + \frac{5}{3} x_3 + \frac{2}{3} x_5$

$\not{0}$ → $x_6 = \frac{20}{3} + \frac{7}{3} x_2 - \frac{2}{3} x_3 + \frac{1}{3} x_5$.

min is 1.

x_2 entre, x_4 leave

→ Current feasible basis $B = \{1, 2, 6\}$ with sol $(4, 1, 0, 0, 0, 9)^T$

$z = 7 + \frac{1}{2} x_3 - \frac{11}{10} x_4 + \frac{2}{5} x_5$

$\not{0}$ → $x_1 = 4 \qquad - \frac{1}{5} x_4 - \frac{1}{5} x_5$

$\not{0}$ → $x_2 = 1 + \frac{1}{2} x_3 - \frac{3}{10} x_4 + \frac{1}{5} x_5$

$\not{0}$ → $x_6 = 9 + \frac{1}{2} x_3 - \frac{7}{10} x_4 + \frac{4}{5} x_5$

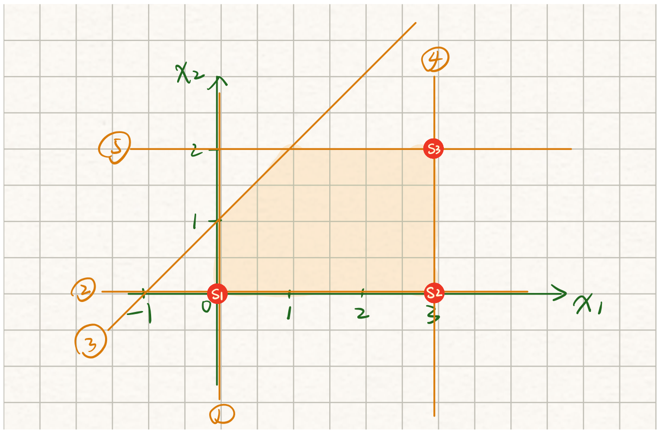
x_3 entre

∴ The LP is unbounded.

The certificate of unboundedness is $\bar{x} = (4, 1, 0, 0, 0, 9)^T$ $d = (0, \frac{1}{2}, 1, 0, 0, \frac{1}{2})^T$

Q. Solve

$$\max\ x_1 + x_2$$
$$\text{s.t.}\quad -x_1 + x_2 \le 1 \quad ③$$
$$x_1 \le 3 \quad ④$$
$$x_2 \le 2 \quad ⑤$$
$$① \ x_1 \geq 0 \qquad ② \ x_2 \geq 0$$



The original LP is SEF:

$$\max \begin{pmatrix} 1 & 1 & \boxed{0 \quad 0 \quad 0} \end{pmatrix} x$$

$$\text{subject to} \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

$$x \geqslant 0.$$

canonical form is $B = \{3, 4, 5\}$

→ The current feasible is $B = \{3, 4, 5\}$ with sol $(0, 0, 1, 3, 2)^T$

$$z = 1x_1 + 1x_2$$

$$\cancel{0} \rightarrow x_3 = 1 + 1 \cdot x_1$$
$$\underset{\triangle}{3} \rightarrow x_4 = 3 - 1 \cdot x_1$$
$$\cancel{0} \rightarrow x_5 = 2 - 1x_2$$

x_1 entre. x_4 leave → $x_1 = 3 - x_4$ 代入

→ The current feasible is $B = \{1, 3, 5\}$ with sol $(3, 0, 4, 0, 2)^T$

$$z = 3 + x_2 - x_4$$

$$\cancel{0} \rightarrow x_1 = 3 - x_4$$
$$4 \rightarrow x_3 = 4 - x_2 - x_4$$
$$\underset{\triangle}{2} \rightarrow x_5 = 2 - x_2$$

min: 2

x_2 entre. x_5 leave → $x_2 = 2 - x_5$ 代入

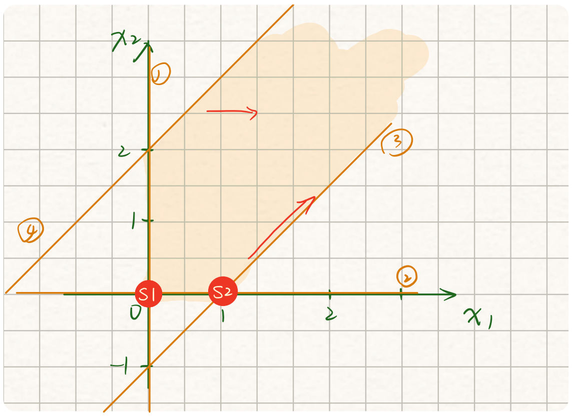
→ The current feasible basis is $B = \{1, 2, 3\}$ .with sol $(3, 2, 2, 0, 0)^T$

$$z = 5 - x_4 - x_5$$

$$x_1 = 3 - x_4$$
$$x_2 = 2 - 1 \cdot x_5$$
$$x_3 = 2 - x_4 + 1 \cdot x_5$$

Q. Solve

$$\begin{array}{ll} \max & x_1 \\ \text{s.t.} & x_1 - x_2 \le 1 \quad ③ \\ & -x_1 + x_2 \le 2 \quad ④ \\ & x_1 \ge 0 \; ① \qquad x_2 \ge 0 \; ② \end{array}$$



The current SEF is:

$$\max \begin{pmatrix} 1 & 0 & \boxed{0 \quad 0} \end{pmatrix} x$$

$$\text{s.t} \begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$x \ge 0$$

canonical form: $B = \{3, 4\}$

→ The current feasible is $B = \{3, 4\}$ with sol $(0, 0, 1, 2)^T$

$$z = x_1$$

$1 \rightarrow x_3 = 1 - x_1 + x_2$

$\emptyset \rightarrow x_4 = 2 + x_1 - x_2$

x_1 entre. x_3 leave

→ The current feasible is $B = \{1, 4\}$ with sol $(1, 0, 0, 3)^T$

$$z = 1 + 1 \cdot x_2 - x_3$$

$$x_1 = 1 + 1 \cdot x_2 - x_3$$
$$x_4 = 3 \qquad - x_3$$

unbounded: $d = (1, 1, 0, 0)^T$

$$\bar{x}_1(t) = 1 + 1 \cdot t$$
$$\bar{x}_2(t) = 1 \cdot t$$
$$\bar{x}_3(t) = 0$$
$$\bar{x}_4(t) = 3 + 0 \cdot t$$

Two Phase Simplex method

$$\max \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} x$$

$$\text{s.t} \begin{pmatrix} -1 & -2 & 1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$x \geqslant 0$$

Phase I.

$$\max \begin{pmatrix} 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

$$\text{s.t} \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$x \geqslant 0$$

→ basis $B = \{4, 5\}$. sol $\bar{x} = (0\ 0\ 0\ 1\ 3)^T$

$$z = -4 + 2x_1 + x_2$$

$1 \rightarrow x_4 = 1 - x_1 - 2x_2 + x_3$

$3 \rightarrow x_5 = 3 - x_1 + x_2 - x_3$

↓

min 1. x_4 leave x_1 entre.

→ basis $B = \{1, 5\}$. sol $\bar{x} = (1\ 0\ 0\ 0\ 2)^T$

$$z = -2 - 3x_2 \underline{+ 2}x_3 - 2x_4$$

$\emptyset \rightarrow x_1 = 1 - 2x_2 + x_3 - x_4$

$1 \rightarrow x_5 = 2 + 3x_2 - 2x_3 + x_4$

↓

min 1 x_5 leave x_3 entre.

→ basis $B = \{1, 3\}$. sol $\bar{x} = (2\ 0\ 1\ 0\ 0)^T$

$$z = -x_4 - x_5 \longleftarrow \text{STOP.}$$

$$x_1 = 2 - \tfrac{1}{2}x_2 - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5$$

$$x_3 = 1 + \tfrac{3}{2}x_2 + \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5$$

∴ Phase I 证得 P 有解

Phase II

PROPOSITION 2.4 *Suppose an LP*

$$\max\{z(x) = c^{\top}x + \bar{z} : Ax = b, x \geq \mathbf{0}\}$$

and a basis B of A are given. Then the following LP is an equivalent LP in canonical form for the basis B:

$$\begin{aligned} \max \quad & z(x) = y^{\top}b + \bar{z} + (c^{\top} - y^{\top}A)x \\ \textit{subject to} \quad & \\ & A_B^{-1}Ax = A_B^{-1}b \\ & x \geq 0, \end{aligned}$$

where $y = A_B^{-T}c_B$.

$$\boxed{\begin{aligned} \max \quad & \begin{pmatrix} 0 & 0 & 0 & -1 & -1 \end{pmatrix} \\ \text{s.t} \quad & \begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{3}{2} & 1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}}$$

$\rightarrow B = \{1, 3\}$ $\quad y = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-T} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$$z = \begin{pmatrix} 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \left(\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix}^{T} \begin{pmatrix} -1 & -2 & 1 \\ 1 & -1 & 1 \end{pmatrix} \right) x$$
$$= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} + 6$$

$$\boxed{\begin{aligned} \max \quad & \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} x + 6 \\ \text{s.t.} \quad & \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x \geq 0 \end{aligned}}$$

$\rightarrow B = \{2, 3\}$

$$\boxed{\begin{aligned} \max \quad & \begin{pmatrix} -2 & 0 & 0 \end{pmatrix} x + 10 \\ \text{s.t} \quad & \begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \\ & x \geq 0 \end{aligned}}$$

$\therefore (0, 4, 7)^T$ is an opt sol

Q. Compute certificate of unboundedness if Simplex Method stopped at

$$\boxed{\begin{array}{rcl} z &=& 27 - 1x_1 + 2x_2 - 1x_3 \\ \\ x_4 &=& 3 - 2x_1 + 1x_2 - 3x_3 \\ x_5 &=& 0 + 1x_1 \qquad\quad + 1x_3 \\ x_6 &=& 3 + 6x_1 + 2x_2 - 1x_3 \end{array}}$$

the current feasible basis $B = \{4, 5, 6\}$

→ x_2 entres the basis, but there is no leaving variables

$$\bar{x} = (0 \quad 0 \quad 0 \quad 3 \quad 0 \quad 3)^T$$

$$d = (\underbrace{0 \quad 1 \quad 0}_{N} \quad \underbrace{1 \quad 0 \quad 2}_{B})^T$$

There is no entring variable but NO leaving, then the LP is unbounded.

* Simple Method One Phase

If during Simplex Method, there is NO entering variables.

then we are at an optimal sol.

* Simplex Method Two Phase

Solve

$$\boxed{\begin{array}{l} \max \ (-3 \quad 0 \quad 1 \quad 0)\, x \\ \text{s.t.} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & 1 & -1 & 0 \\ 0 & 3 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 1 \\ 9 \end{pmatrix} \\ \qquad x \geq 0 \end{array}} \qquad (P)$$

Step 1.

→ Add auxiliary variables $x_5 . x_6 . x_7$

$$\max \ \underbrace{(0 \quad 0 \quad 0 \quad 0 \quad -1 \quad -1 \quad -1)}\, x \rightarrow \text{penalty}$$

$$\text{s.t.} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -2 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 1 \\ 9 \end{pmatrix} \qquad (Q)$$

$$x \geq 0$$

* Q is not infeasible. $B=\{5, 6, 7\}$ is a feasible basis with basic feasible sol $\bar{x}=(0\ 0\ 0\ 0\ 4\ 1\ 9)^T$

* Q is not unbounded. for every feasible x for Q. $value_Q(x) = -x_5 - x_6 - x_7 \le 0$
$\therefore value_Q(x) \le 0$

* Q has optimal sol. $\tilde{x}$. $(\tilde{x} \geq 0)$

$value_Q(\tilde{x}) = 0 \iff \tilde{x}_5 = \tilde{x}_6 = \tilde{x}_7 = 0.$

$\iff (\tilde{x}_1 . \tilde{x}_2 . \tilde{x}_3 . \tilde{x}_4)^T$ is feasible for P.

$$\begin{cases} value_Q(x) < 0 & P \text{ infeasible} \\ \underset{\uparrow \atop \text{optimal value}}{value_Q(x) = 0} & P \text{ feasible} \end{cases}$$

→ The current feasible basis $B=\{5, 6, 7\}$. with basic sol $\bar{x}=(0\ 0\ 0\ 0\ 4\ 1\ 9)^T$

$z = -x_5 - x_6 - x_7 = -14 - x_1 + 5x_2 + x_3 + 2x_4$

代入

$x_5 = 4 - x_1 - x_2 - x_3 - x_4$ $\qquad x_2 = 4 - x_1 - x_3 - x_4 - x_5$
$x_6 = 1 + 2x_1 - x_2 + x_3$
$x_7 = 9 \quad -3x_2 - x_3 - x_4$

x_2 entre. x_6 leave.

→ The current feasible basis $B=\{2, 5, 7\}$. with feasible basic sol $\bar{x}=(0\ 1\ 0\ 0\ 3\ 0\ 6)^T$

$z = -9 + 9x_1 + 6x_3 + 2x_4 - 5x_6$

$\phi \to x_2 = 1 + 2x_1 + 1x_3 \quad - x_6$
$1 \to x_5 = 3 - 3x_1 - 2x_3 - 1x_4$ $\qquad$ 上面 $x_5 - x_6$
$1 \to x_7 = \underset{\sim}{6} - 6x_1 - 4x_3 - 1x_4 + 3x_6$

min is 1 $\qquad$ X_1 entres. x_5 leaves.

→ The current feasible basis $B=\{1, 2, 7\}$. with feasible basic sol $\bar{x}=(1\ 3\ 0\ 0\ 0\ 0\ 0)^T$

$z = 0 \quad -1x_4 \quad -3x_5 \quad -2x_6$

$x_1 = 1 - \frac{2}{3}x_3 - \frac{1}{3}x_4 - \frac{1}{3}x_5 + \frac{1}{3}x_6$
$x_2 = 3 - \frac{1}{3}x_3 - \frac{2}{3}x_4 - \frac{2}{3}x_5 - \frac{1}{3}x_6$
$x_7 = \quad 1x_4 + 2x_5 + 1x_6$

P is feasible and both $\{1, 2, 3\}$ and $\{1, 2, 4\}$ are feasible b
because $\bar{x}=(1, 3, 0, 0)^T$. is feasible sol for P.

Step 2.

→ The current feasible basis $B = \{1, 2, 4\}$ with basic feasible sol $\bar{x} = (1\ 3\ 0\ 0)^T$
with basic feasible solution $\bar{x} = (1\ 3\ 0\ 0)^T$

$$z = -3 + 3x_3$$

$$\left.\begin{aligned} x_1 &= 1 - \tfrac{2}{3}x_3 \\ x_2 &= 3 - \tfrac{1}{3}x_3 \\ x_4 &= 0 \end{aligned}\right\} \text{can be solved from Step 1.}$$

x_3 entres. x_1 leaves

→ The current feasible basis $B = \{2, 3, 4\}$. with basic feasible sol $\bar{x} = (0\ \tfrac{5}{2}\ \tfrac{3}{2}\ 0)^T$

$$z = \tfrac{3}{2} - \tfrac{9}{2}x_1$$

$$\begin{aligned} x_3 &= \tfrac{3}{2} - \tfrac{3}{2}x_1 \\ x_2 &= \tfrac{5}{2} + \tfrac{1}{2}x_1 \\ x_4 &= 0 \end{aligned}$$

no entring

→ The certificate of optinality can be computed as: (proposition 2.4. page 61)

$$y = A_B^{-T} c_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix}$$

→ Geometry

- def. Hyperphere 超球体 & Halfspace 半空间

Given $a \in \mathbb{R}^d$. $a \neq 0$. $\beta \in \mathbb{R}$

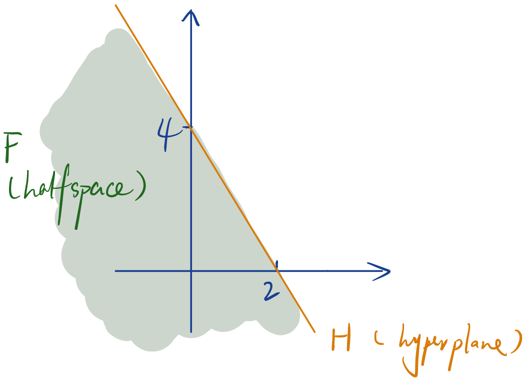
$a^T b = \|a\| \|b\| \cos\theta$

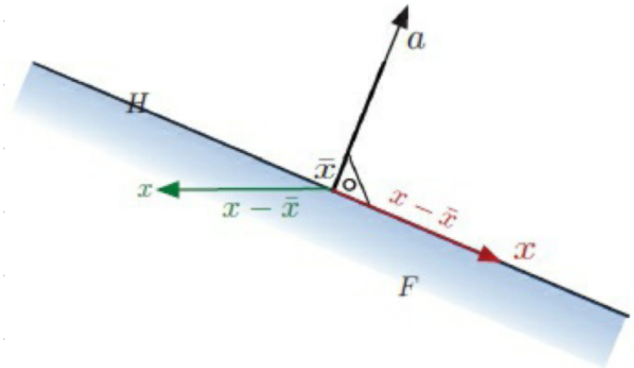
hyperphere $H := \{x \in \mathbb{R}^d, a^T x = \beta\}$

half space $F := \{x \in \mathbb{R}^d, a^T x \le \beta\}$

H: set of points which $a \perp x - \bar{x}$

F: set of points which a & $x - \bar{x}$ 夹角 90°





$a = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\beta = 4$

$H = \{x \in \mathbb{R}^2 : (2 \quad 1)\, x = 4\}$ $\Rightarrow 2x1 + 1x2 = 4$

$F = \{x \in \mathbb{R}^2 : (2 \quad 1)\, x \le 4\}$ $\Rightarrow 2x1 + 1x2 \le 4$

Theorem 6: Dimension of Hyperplane

The dimension of a hyperplane in $\mathbb{R}^n$ is $n - 1$.

proof:

For $a \in \mathbb{F}^n$ $\dim\{x : a^T x = 0\} + \operatorname{rank}(a^T) = n$

$\because a \neq 0$ by definition, $\dim(a^T) = 1$

$\therefore \dim(H) = \dim\{x : a^T x = 0\} = n - 1$

- Polyerhedron 多面体

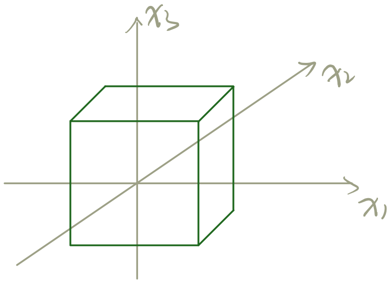
a set of P s.t $P = \{x \in \mathbb{R}^d : Ax \le b\}$

* a polyerhedron is the intersection of a finite number of halfspace

ep. cube is a polyledron

$$P = \{x \in \mathbb{R}^d : 0 \le x_1 \le 1 ,\ 0 \le x_2 \le 1 ,\ 0 \le x_3 \le 1\}$$

$$= \left\{ x \in \mathbb{R}^3 : \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} x \le \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$



Proposition :

The feasible region of an LP is a polyhedron

(a finite number of halfspaces)

$$a^T x \ge \beta \iff -a^T x \le -\beta$$

$$a^T x = \beta \iff \begin{cases} -a^T x \le -\beta \\ a^T x \ge \beta \end{cases}$$

ep. max $x_1 + 2x_2$

s.t. $x_1 + 2x_2 \ge 4$

$-x_1 + x_2 \ge -4$

$x_1 + x_2 \le 10$

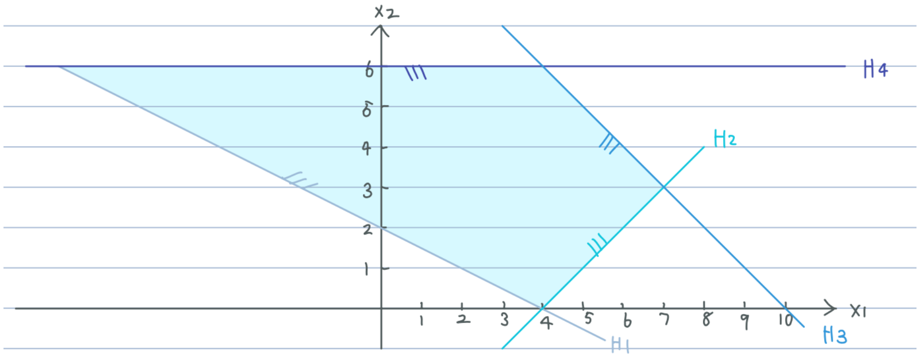
$x_2 \le 6$

$H_1 = \{x \in \mathbb{R} : (-1\ -2)\ x \le 4$ $\qquad x_2 \ge 2 - \frac{1}{2}x_1$

$H_2 = \{x \in \mathbb{R} : (1\ -1)\ x \le 4$ $\qquad x_2 \ge -4 + x_1$

$H_3 = \{x \in \mathbb{R} : (1\ \ 1)\ x \le 10$ $\qquad x_2 \le 10 - x_1$

$H_4 = \{x \in \mathbb{R} : (0\ \ 1)\ x \le 6$ $\qquad x_2 \le 6$

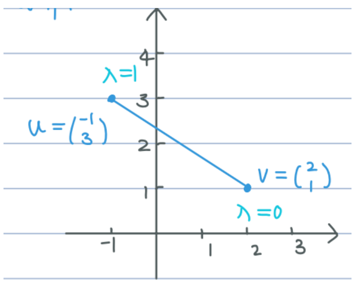


- line segment

Given 2 points $u, v \in \mathbb{R}^d$, the line segment between u & v is:

$$\{x \in \mathbb{R}^d : x = \lambda u + (1-\lambda) v \quad 0 \le \lambda \le 1\}$$

线段：由2点间所有的点组成



ep. $u = \binom{-1}{3}$ $\quad v = \binom{2}{1}$

the line segment between u & v:

$$\left\{x \in \mathbb{R}^2 : x = \begin{pmatrix} -\lambda + (1-\lambda)\cdot 2 \\ 3\lambda + (1-\lambda)\cdot 1 \end{pmatrix}, \ 0 \le \lambda \le 1\right\}$$

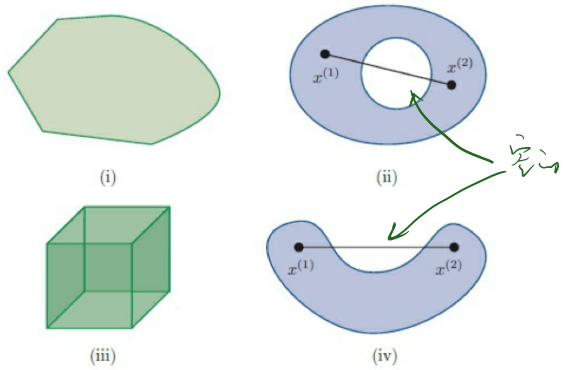
- convex

一个饱满的形状

a set $S \in \mathbb{R}^d$ is called convex if for every pair of points $u, v \in S$. the lines between u & v lies in S.

$$\forall u, v \in S.\ \lambda \in [0,1].\quad \lambda u + (1-\lambda) v \in S$$

ep.



(i) (ii)

(iii) (iv)

convex not convex

poposition: every half space is a convex set

proof: Let $H := \{x \in \mathbb{R}^d, a^T x \le \beta\}$ be a halfspace. $a \in \mathbb{R}^d$. $a \ne 0$. $\beta \in \mathbb{R}$

show $\forall u, v \in H.\ \lambda u + (1-\lambda) v \in H$:

$$a^T(\lambda u + (1-\lambda) v) = \underbrace{\lambda}_{\ge 0} \underbrace{a^T u}_{\le \beta} + \underbrace{(1-\lambda)}_{\ge 0} \underbrace{a^T v}_{\le \beta} \le \lambda \beta + (1-\lambda)\beta = \beta$$

$\therefore \lambda u + (1-\lambda) v \in H$

poposition: Given a convex sets $c_j \in \mathbb{R}^d$, $j \in J$. The intersection $C := \bigcap \{C_j : j \in J\}$ is also a convex set

(J can be ∞)

proof: show $\forall u, v \in C$, $\lambda \in [0,1]$. $\lambda u + (1-\lambda) v \in C$:

$\because u, v \in C$. $\therefore u, v \in c_j \quad \forall j \in J$

$\because c_j$ is convex. $u, v \in c_j$. $\forall j$. $\therefore \lambda u + (1-\lambda) v \in c_j \quad \forall j$

$\because C = \bigcap_{j \in J} c_j \qquad \therefore \lambda u + (1-\lambda) v \in C$

theorem: every polyhedron is a convex set

proof: 依据前面的 poposition

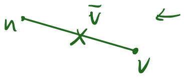
- Extreme points

表面上的点

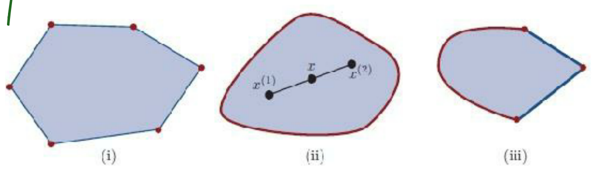
Let $S \subseteq \mathbb{R}^d$ be a convex set. A point $\bar{x} \in S$ is extreme point if:

$\exists$ no $u, v \in S$, $\lambda \in [0,1]$ s.t

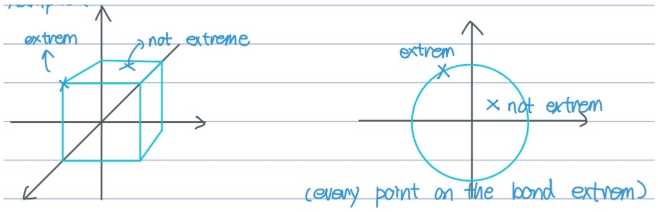
1. $u \neq v$
2. $\lambda \neq 1 \wedge \lambda \neq 0 \qquad (0 < \lambda < 1)$
3. $\bar{x} = \lambda \cdot u + (1-\lambda) v$.

不能被夹在 uv 之间  ← 不是 extreme point

eg.



(i) (ii) (iii)



(every point on the bond extrem)

proposition: Let $P = \{x \in \mathbb{R}^d : Ax \le b\}$ be a polyhedron, $\bar{x} \in P$.

Let $A^= x = b^=$ be the set of tight constraints for $\bar{x}$.

Then $\bar{x}$ is an extreme point of P iff $\operatorname{rank}(A^=) = d \rightarrow$ dimension$(\mathbb{R}^d)$

✓ $Ax = b$
× $Ax \le b$

Proof Idea for Proposition 9 $\uparrow$

$\Leftarrow$

If $\operatorname{rank}(A^{=}) = d$, but $\bar{x}$ is not a extreme point.

So there are $u \in P, v \in P$, $\lambda \in [0,1]$ such that $u \neq v$, $\lambda \neq 0$, $\lambda \neq 1$ and $\bar{x}$ is still $\bar{x} = \lambda u + (1-\lambda) v$

We have $b^{=} = A^{=}\bar{x} = A^{=}(\lambda u + (1-\lambda) v) = A^{=}\lambda u + A^{=}(1-\lambda) v = \underbrace{\lambda}_{\geqslant 0}\underbrace{(A^{=}u)}_{\leqslant b^{=}} + \underbrace{(1-\lambda)}_{\geqslant 0}\underbrace{(A^{=}v)}_{\leqslant b^{=}} \leqslant (\lambda + 1 - \lambda) b^{=} = b^{=}$ 有点 $x^2+y^2=0 \Rightarrow x=y=0$

So $\lambda(A^{=}u) = \lambda b^{=}$ and $(1-\lambda)(A^{=}v) = (1-\lambda) b^{=}$

So $A^{=}u = b^{=} = A^{=}v$ since $\lambda \neq 0$, $1-\lambda \neq 0$

Since $\operatorname{rank}(A^{=}) = d$, we have unique solution for $A^{=}x = b^{=}$

Thus $u = v$, contradict

$\Rightarrow$

If $\bar{x}$ is a extreme point, but $\operatorname{rank}(A^{=}) \neq d \Rightarrow \operatorname{rank}(A) < d$

So there exist non-zero $y \in \mathbb{R}^d$ such that $A^{=}y = \emptyset$

Consider $u = \bar{x} + \varepsilon \cdot y$, $v = \bar{x} - \varepsilon \cdot y$ for small enough $\varepsilon > 0$

for a^T in A

$A^{=}u = A^{=}\bar{x} + \varepsilon A^{=}y = A^{=}\bar{x} + \emptyset = b^{=}$, $A^{=}v = A^{=}\bar{x} - \varepsilon A^{=}y = A^{=}\bar{x} - \emptyset = b^{=}$

for a^T not in A

$a^T u = a^T\bar{x} + \varepsilon a^T y < \beta + \varepsilon a^T y$ (这玩意足够小) $\leqslant \beta$, $a^T v = a^T\bar{x} - \varepsilon a^T y < \beta - \varepsilon a^T y$ (这玩意足够小) $\leqslant \beta$

$\}$ $u \in P$ $v \in P$

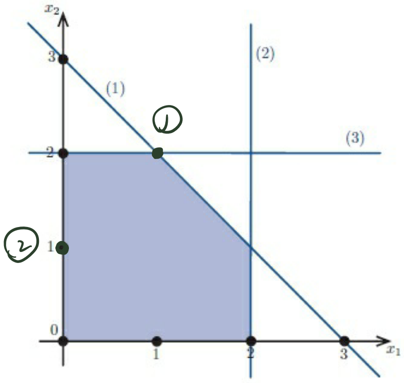
So $u \in P$, $v \in P$, $\lambda = \frac{1}{2}$, $\bar{x} = \frac{1}{2}u + \frac{1}{2}v = \frac{1}{2}(\bar{x} + \varepsilon y) + \frac{1}{2}(\bar{x} - \varepsilon y)$

Thus $\bar{x}$ is not a extreme point, contradict

Q. max $(c_1, c_2)x$

s.t.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \le \begin{pmatrix} 3 \\ 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}. \quad \begin{array}{l} ✓ (1) \\ (2) \\ ✓ (3) \\ (4) \\ (5) \end{array}$$



判断 ① & ② 是否是 extreme point.

① $\bar{x} = (1 \quad 2)^T$

$Ax = (3 \quad 1 \quad 2 \quad -1 \quad -2)^T \quad \therefore (1)(3)$ tight

$A^= = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\text{rank}(A^=) = 2 = d$ extreme

② $\bar{x} = (0 \quad 1)^T$

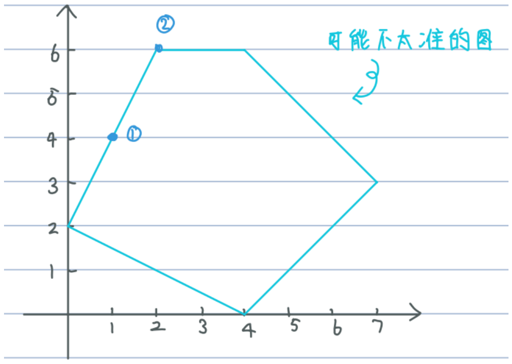
$Ax = (1 \quad 0 \quad 1 \quad 0 \quad -1)^T \quad \therefore (4)$ tight

$A^= = [-1 \quad 0]$

$\text{rank}(A^=) = 1 \neq d$ not extrem

Q.

$$\left\{ x \in \mathbb{R}^2 : \begin{pmatrix} -1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 0 & 1 \\ -2 & 1 \end{pmatrix} x \le \begin{pmatrix} -4 \\ 4 \\ 10 \\ 6 \\ 2 \end{pmatrix} \right\} \quad \begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \\ (5) \end{array}$$



可能不太准的图

判断 ① & ② 是否是 extreme point.

① $\bar{x} = (1 \quad 4)^T$

$Ax = (-9 \quad -3 \quad 5 \quad 4 \quad 2)^T \quad \therefore (5)$ tight

$A^= = [-2 \quad 1]$

$\text{rank}(A^=) = 2 = d$ extrem

② $\bar{x} = (2 \quad 6)^T$

$Ax = (-14 \quad -4 \quad 8 \quad 6 \quad 2)^T \quad \therefore (4)(5)$ tight

$A^= = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$

$\text{rank}(A^=) = 2 = d$ extrem

proposition: Let $P=\{x\in\mathbb{R}^d : Ax=b.\ x\geq 0\}$ be a polyhedron. $\bar{x}\in P$.

Then $\bar{x}$ is an extrem point of P $\Leftrightarrow$ $\bar{x}$ is a basic feasible sol $Ax=b$.

proof: $\bar{B}=\{i : i=1,\ldots,d,\ \bar{x}_i\neq 0\}$ $\quad\bar{N}=\{i=1,\ldots,d.\ \bar{x}_i=0\}$.

$\bar{x}$ is an extreme point of P.

$\Leftrightarrow$ $\bar{x}$ is feasible. $\operatorname{rank}(A^{=}) = \operatorname{rank}\left(\begin{bmatrix} A_{\bar{N}} & A_{\bar{B}} \\ -A_{\bar{N}} & -A_{\bar{B}} \\ -I & 0 \end{bmatrix}\right)=d$

$\Leftrightarrow$ $\bar{x}$ is feasible. $A_{\bar{B}}$ has linearly independent cols

$\Leftrightarrow$ $\bar{x}$ is basic feasible sol.

Q. $P=\left\{x\in\mathbb{R}^4,\ \begin{bmatrix} 1 & 3 & 14 & -7 \\ 0 & 2 & 8 & -4 \end{bmatrix} x = \begin{bmatrix} 7 \\ 4 \end{bmatrix},\ x\geq 0\right\}$

determine whether $\bar{x}=(1\ 2\ 0\ 0)^T$ and $\bar{x}=(0\ 0\ 1\ 1)^T$ are extreme point

$P=\left\{x\in\mathbb{R}^4,\ \begin{bmatrix} 1 & 3 & 14 & -7 \\ 0 & 2 & 8 & -4 \end{bmatrix} x = \begin{bmatrix} 7 \\ 4 \end{bmatrix},\ x\geq 0\right\}$

$$P=\left\{x\in\mathbb{R}^4;\ \begin{bmatrix} 1 & 3 & 14 & -7 \\ 0 & 2 & 8 & -4 \\ -1 & -3 & -14 & 7 \\ 0 & -2 & -8 & 4 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} x \leq \begin{bmatrix} 7 \\ 4 \\ -7 \\ -4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ x\geq 0\right\}$$

找 rows 符合

$A^{=}x=b^{=}$

$\bar{x}=(1\ 2\ 0\ 0)^T$ $\qquad$ $\bar{x}=(0\ 0\ 1\ 1)^T$

$$A^{=}=\begin{bmatrix} 1 & 3 & 14 & -7 \\ 0 & 2 & 8 & -4 \\ -1 & -3 & -14 & 7 \\ 0 & -2 & -8 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad A^{=}=\begin{bmatrix} 1 & 3 & 14 & -7 \\ 0 & 2 & 8 & -4 \\ -1 & -3 & -14 & 7 \\ 0 & -2 & -8 & 4 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$\bar{x}$ lies on 6 hyperplanes

6 hyperplane 交于$\underset{\bar{x}}{\underline{一点}}$ $\Leftrightarrow$ $A^{=}$ has full-rank

$(\operatorname{rank}(A^{=})=4)$

line segement

$\{x \in \mathbb{R} : \lambda u + (1-\lambda) u ,\ 0 \leq \lambda \leq 1\}$

convex

set S. s.t.

$\forall u \in S, v \in S . \lambda \in [0,1]$

$\lambda u + (1-\lambda) v \in S$

判断 convex

extreme point

$\bar{x} \in S$ s.t. $\forall u . v . \lambda$

① $u \neq v$
② $\lambda \in (0,1)$
③ $\bar{x} = \lambda u + (1-\lambda) v$.

↑
3个条件无法满足 IMPOSSIBLE.

判断 extre

proposition ①

every half space is a convex

+

proposition ②

every intersection of convex is a convex

⇓

proposition ③

every polyhedron is a convex

proposition ④

$P = \{x \in \mathbb{R}^d : Ax \leq b\}$

$x \in P$ is extreme $\Leftrightarrow$ $\text{rank}(A^{=}) = d$

proposition ⑤

$P = \{x \in \mathbb{R}^d : Ax = b,\ x \geq 0\}$

$x \in P$ is extreme $\Leftrightarrow$ x is basic feasible sol

hyperplane

$a \in \mathbb{R}^d .\ a \neq 0 .\ \beta \in \mathbb{R} .$

$H : \{x \in \mathbb{R}^d : a^T x = \beta\}$

超平面

$\mathbb{R}^2$ 中. hyperplane 为一条直线

$\mathbb{R}^3$ 中. hyperplane 为一个平面

"分割空间"

half space

$a \in \mathbb{R}^d .\ a \neq 0 .\ \beta \in \mathbb{R}$

$H : \{x \in \mathbb{R}^d : a^T x \leq \beta\}$

半空间

由超平面分割后的区域

(超平面的一侧, 包括超平面).

polyhedron

$A \in M_{m \times d}(\mathbb{R})\quad b \in \mathbb{F}^m$

$P : \{x \in \mathbb{R}^d : Ax \leq b\}$

多面体

有限个面交集组成的多维形状

4. Duality

- Weak duality

Q.

$$\max \ (3 \ \ 2 \ \ 4)\, x$$
$$\text{s.t.} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 2 & 1 & 3 \end{pmatrix} x \le \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$$
$$x \ge 0$$

自己凑数

feasible sol	value
$(0 \ 0 \ 0)^T$	0
$(1 \ 1 \ 1)^T$	9
$(2 \ 2 \ 0)^T$	10
$(\frac{5}{2} \ \frac{3}{2} \ 0)^T$	10.5

GUESS

How can we produce an upper bound on the optimal value of the LP?

→ multiply constraints by $y_1 . y_2 . y_3$ $(y_1 . y_2 . y_3 \ge 0)$

$$(y_1 \ y_2 \ y_3)\begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} \ge (y_1 \ y_2 \ y_3)\begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 2 & 1 & 3 \end{pmatrix} x$$

Guess. $\bar{y} = (1 \ 1 \ 1)^T$

$16 \ge (5 \ 2 \ 8)\, x$ holds $\forall x$ feasible

$$y^T b \qquad \bar{y}^T\begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} \qquad y^T A \qquad \bar{y}^T\begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$(3 \ 2 \ 4)\, x \le (3 \ 2 \ 4) + (\underline{16} - \underline{(5 \ 2 \ 8)}\, x)$$

$$= 16 + \underbrace{(-2 \ \ 0 \ \ -4)}_{\le 0}\underbrace{x}_{\ge 0} \le 16$$

→ To obtain best possible such bound, we can write the following LP

$$y^T A \ge c^T$$
$$y^T \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 2 & 1 & 3 \end{pmatrix} \ge (3 \ \ 2 \ \ 4)$$

让 $\bar{y}^T A - c^T$ 的符号统一

使我们能比较大

$\Rightarrow$

$$\min \ (4 \ \ 5 \ \ 7)\, y$$

代表 upper bound. 找最小

$$A^T y \ge c$$

$$\text{s.t} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 3 & 3 \end{pmatrix} y \ge \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

$$y \ge 0$$

为了不让 duality 翻转

feasible sol	value
$(0 \ 0 \ 0)^T$	0
$(1 \ 1 \ 1)^T$	9
$(2 \ 2 \ 0)^T$	10
$(\frac{5}{2} \ \frac{3}{2} \ 0)^T$	10.5

上升

val ≥ 10.5 → val = 10.5 ← val ≤ 10.5

feasible sol	value
$(1, 1, 1)^T$	16
$(2, 0, 1)^T$	15
$(3, 0, 0)^T$	12
$(2, \frac{1}{2}, 0)^T$	10.5

下降

– Theorem : Weak duality , Special Case

consider the LPs

Ⓟ
$$\boxed{\begin{array}{ll} \max & c^T x \\ \text{s.t} & Ax \leq b \\ & x \geq 0 \end{array}}$$

$$\boxed{\begin{array}{ll} \min & b^T y \\ \text{s.t.} & A^T y \geq c \\ & y \geq 0 \end{array}}$$
Ⓓ

Let $\bar{x}$ be feasible for (P) . $\bar{y}$ feasible for (D)

then 1) $c^T \bar{x} \leq b^T \bar{y}$

2) if $c^T \bar{x} = b^T \bar{y}$, then $\bar{x}$ optimal for (P) & $\bar{y}$ optimal for (D)

proof :

1) Let x be a feasible sol for (P), So $Ax \leq b$ $x \geq 0$

$\because$ $\bar{y}$ is feasible for (D). $\bar{y} \geq 0$.

$\therefore$ $y^T A x \leq y^T b$

$$c^T x \leq c^T x + (\bar{y}^T b - \bar{y}^T A x)$$
$$= y^T b + (c^T x - \bar{y}^T A x)$$
$$= y^T b + \underbrace{(c^T - y^T A)}_{\substack{\leq 0 \\ (A^T \bar{y} \leq c)}} \underbrace{x}_{\substack{\geq 0 \\ (x \text{ is feasible for } P)}}$$
$$\leq \bar{y}^T b = b^T y$$

$\therefore$ $c^T x \leq b^T \bar{y}$ $\forall$ x feasible for (P)

2) if $c^T \bar{x} = b^T \bar{y}$, then $c^T x \leq b^T \bar{y} = c^T \bar{x}$ $\forall x$ feasible for (P)

$\therefore$ $\bar{x}$ is optimal for (P)

$\bar{y}$ 同理

- Theorem: Weak duality, SEF case

consider the LPs

(P)
$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

(D)
$$\begin{array}{ll} \min & b^T y \\ \text{s.t.} & A^T y \geq c \\ & y \text{ free} \end{array}$$

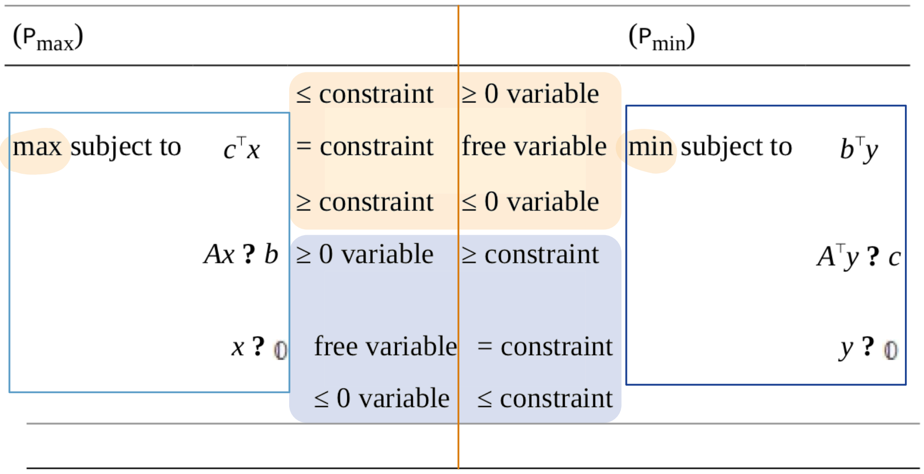
Let $\bar{x}$ be a feasible solution for (P)
$\bar{y}$ be a feasible solution for (D)

1) $c^T\bar{x} \leq b^T\bar{y}$

2) if $c^T\bar{x} = b^T\bar{y}$. then $\bar{x}$ is optimal for (P)
$\bar{y}$ is optimal for (D)

* construct dual LPs .

Table 4.1 Primal–dual pairs



(P_{max})			(P_{min})
	≤ constraint	≥ 0 variable	
max subject to c^Tx	= constraint	free variable	min subject to b^Ty
	≥ constraint	≤ 0 variable	
Ax **?** b	≥ 0 variable	≥ constraint	A^Ty **?** c
x **?** 0	free variable	= constraint	y **?** 0
	≤ 0 variable	≤ constraint	

$$\begin{tikzcd} \text{constraints } \langle A_1, x\rangle = b_1 \arrow[dr] & A_1^T y \geq c_1 \\ \text{variables } x_1 \geq 0 \arrow[ur] & y_1 \text{ free} \end{tikzcd}$$

Q. Construct dual for the following LP.

(P)

$$\underline{\underline{\max}} \quad \begin{pmatrix} 3 & 5 & 10 \end{pmatrix} x$$

$$\text{s.t.} \quad \begin{pmatrix} 4 & 1 & 0 \\ 3 & 4 & 3 \\ 0 & 1 & 7 \\ 2 & 0 & 1 \end{pmatrix} x \quad \begin{matrix} \geq \\ \leq \\ = \\ \leq \end{matrix} \quad \begin{pmatrix} 1 \\ 8 \\ 2 \\ 1 \end{pmatrix}$$

$$x_1 \text{ free} \quad x_2 \geq 0 \quad x_3 \leq 0$$

(D)

$$\min \quad \begin{pmatrix} 1 & 8 & 2 & 1 \end{pmatrix} y$$

$$\text{s.t.} \quad \begin{pmatrix} 4 & 3 & 0 & 2 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 7 & 1 \end{pmatrix} y \quad \begin{matrix} = \\ \boxed{\begin{matrix} \geq \\ \leq \end{matrix}} \end{matrix} \quad \begin{pmatrix} 3 \\ 5 \\ 10 \end{pmatrix}$$

$$\underline{y_1 \leq 0 \quad y_2 \geq 0} \quad y_3 \text{ free} \quad \underline{y_4 \geq 0}$$

(P)

$$\underline{\underline{\min}} \quad \begin{pmatrix} 3 & 5 & 10 \end{pmatrix} x$$

$$\text{s.t.} \quad \begin{pmatrix} 4 & 1 & 0 \\ 3 & 4 & 3 \\ 0 & 1 & 7 \\ 2 & 0 & 1 \end{pmatrix} x \quad \begin{matrix} \geq \\ \leq \\ = \\ \leq \end{matrix} \quad \begin{pmatrix} 1 \\ 8 \\ 2 \\ 1 \end{pmatrix}$$

$$x_1 \text{ free} \quad x_2 \geq 0 \quad x_3 \leq 0$$

(D)

$$\max \quad \begin{pmatrix} 1 & 8 & 2 & 1 \end{pmatrix} y$$

$$\text{s.t.} \quad \begin{pmatrix} 4 & 3 & 0 & 2 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 7 & 1 \end{pmatrix} y \quad \begin{matrix} = \\ \boxed{\begin{matrix} \leq \\ \geq \end{matrix}} \end{matrix} \quad \begin{pmatrix} 3 \\ 5 \\ 10 \end{pmatrix}$$

$$\underline{y_1 \geq 0 . \quad y_2 \leq 0} \, . \quad y_3 \text{ free} \quad \underline{y_3 \leq 0}$$

- Theorem : Weak duality

Let (P) & (D) be a primal dual pair of LPs.

(P) be the min LP . (D) be the max LP.

$\bar{x}$ be feasible for (P). $\bar{y}$ be feasible for (D)

Then 1) $\text{value}_P(\bar{x}) \geq \text{value}_D(\bar{y})$

2) if $\text{value}_P(\bar{x}) = \text{value}_D(\bar{y})$, then $\bar{x}$ is optimal for (P)

$\bar{y}$ is optimal for (D)

Q. Consider

$$\boxed{\begin{aligned} &\max \quad (1 \ \ 2 \ \ 0 \ \ 0)\, x \\ &\text{s.t.} \quad \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 5 \end{pmatrix} \end{aligned}}$$

$$\boxed{\begin{aligned} &\min \quad (8 \ \ 5)\, y \\ &\text{s.t.} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} y \geq \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \end{aligned}}$$

feasible for P	Value	feasible for P	Value
$(0 \ \ 5 \ \ 3 \ \ 0)^T$	10	$(0 \ \ 2)^T$	10

- Strong Duality

- Theorem : Strong Duality

Let (P) and (D) be a primal-dual pair of LPs.

If (P) has an optimal solution $\bar{x}$. then (D) has an optimal solution $\bar{y}$
$\text{value}_P(\bar{x}) = \text{value}_D(\bar{y})$

Proof: [Special Case, SEF]

Assume

(P) $\boxed{\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax=b \\ & x \geq 0 \end{array}}$ $\boxed{\begin{array}{ll} \min & b^T y \\ \text{s.t.} & A^T y \geq c \\ & y \text{ free} \end{array}}$ (D)

assume $\bar{x}$ is optimal for (P)

Let us run simplex method with Blend's Rule on (P)

Simplex Method stop at a optimal sol $\tilde{x}$. (同值不及同解)

So the last dictionary is

$$z = \bar{y}^T b + (c^T - \bar{y}^T A) x$$
$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N.$$

$$\max \quad y^T b + \bar{z} + (c^T - y^T A) x$$
$$A_B^{-1} A x = A_B^{-1} b$$
$$A_B^{-1}(A_B x_B + A_N x_N) = A_B^{-1} b$$

where B is the final base and $y_B = A_B^{-1} c_B$

Show $\bar{y}$ is optimal for (D)

① feasible

negative coefficient $\longrightarrow$ $\boxed{\begin{array}{l} c^T - y^T A \leq 0^T \\ c^T \leq y^T A \\ c \leq A^T \bar{y} \end{array}}$

② optimal

$$\text{value}_P(\tilde{x}) = c^T \tilde{x} = \bar{y}^T b + (c^T - \bar{y}^T A)\tilde{x}$$
$$= (c^T - \bar{y} A)_B \tilde{x}_B = y^T b = \text{value}_D(\bar{y})$$

$\therefore$ By weak duality, $\bar{y}$ is optimal for (D). $\text{value}_P(\bar{x}) = \text{value}_P(\tilde{x}) = \text{value}_D(\bar{y})$

Possible outcomes of Primal-Dual Pairs

P \ D	optimal	unbounded	infeasible
optimal	✓ strong	✗ weak	✗ strong
unbounded	✗ weak	✗ weak	✓ weak
infeasible	✗ strong	✓ weak	✓

↳ 原因： x min · y max

$$c^T x \geqslant b^T y$$

✿ It can be happen that both (P) & (D) are infeasible

max $(0 \quad 1)\, x$

s.t $(1 \quad 0)\, x = -1$

$x \geqslant 0$

infeasible

min $(-1)\, y$

s.t $\begin{bmatrix} 1 \\ 0 \end{bmatrix} y \geqslant \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

y free

infeasible

* construct dual LPs

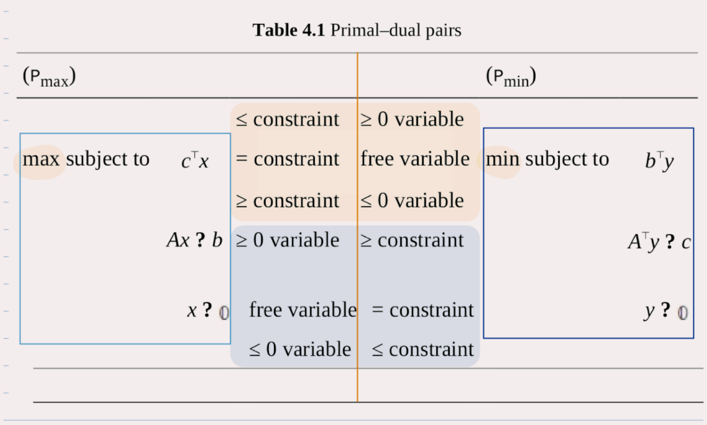


Table 4.1 Primal–dual pairs

$(P_{\max})$			$(P_{\min})$
	≤ constraint	≥ 0 variable	
max subject to $c^\top x$	= constraint	free variable	min subject to $b^\top y$
	≥ constraint	≤ 0 variable	
Ax **?** b	≥ 0 variable	≥ constraint	$A^\top y$ **?** c
x **?** 0	free variable	= constraint	y **?** 0
	≤ 0 variable	≤ constraint	

$$\begin{tikzcd} \text{constraints } \langle A_1, x \rangle = b_1 \arrow[dr] & A_1^T y \geqslant c_1 \\ \text{variables } x_1 \geqslant 0 \arrow[ur] & y_1 \text{ free} \end{tikzcd}$$

– Complementary slackness

Q.

$$\max \quad (4 \quad 1 \quad 5 \quad 3)\, x$$

$$(P) \quad \text{s.t} \quad \begin{array}{c} y_1 \rightarrow \\ y_2 \rightarrow \\ y_3 \rightarrow \end{array} \begin{pmatrix} 1 & -1 & -1 & 3 \\ 5 & 1 & 3 & 8 \\ -1 & 2 & 3 & -5 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 55 \\ 3 \end{pmatrix}$$

$$x_1 . x_2 . x_3 . x_4 \geq 0$$

$$\min \quad (1 \quad 55 \quad 3)\, y$$

$$\text{s.t.} \quad \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{pmatrix} 1 & 5 & -1 \\ -1 & 1 & 2 \\ -1 & 3 & 3 \\ 3 & 8 & -5 \end{pmatrix} y \geq \begin{pmatrix} 4 \\ 1 \\ 5 \\ 3 \end{pmatrix}$$

$$y_1 . y_2 . y_3 \geq 0$$

$\bar{x} = (0 \quad 14 \quad 0 \quad 5)^T$ feasible for (P)

$\bar{y} = (11 \quad 0 \quad 6)^T$ feasible for (D)

$\text{value}_P(\bar{x}) = 14 + 15 = 11 + 18 = \text{value}_D(\bar{y})$

$\therefore$ optimal sol $= 29$ →

当 $\text{value}_P(\bar{x}) = \text{value}_D(\bar{y})$ 时.
$\bar{x}$ 和 $\bar{y}$ 各 optimal sol.

variable in (P)	constant in (D)	variable in (D)	constant in (P)
$\bar{x}_1 = 0$		$\bar{y}_1 = 11 \neq 0$	$\bar{x}_1 - \bar{x}_2 - \bar{x}_3 + 3\bar{x}_4 = 1$ tight
$\bar{x}_2 = 14 \neq 0$	$-1 \cdot \bar{y}_1 + 1 \cdot \bar{y}_2 + 2\bar{y}_3 = 1$ tight	$\bar{y}_2 = 0$	
$\bar{x}_3 = 0$		$\bar{y}_3 = 6 \neq 0$	$-\bar{x}_1 + 2\bar{x}_2 + 3\bar{x}_3 - 5\bar{x}_4 = 3$ tight
$\bar{x}_4 = 5 \neq 0$	$3\bar{y}_1 + 8\bar{y}_2 - 5\bar{y}_3 = 3$ tight		

Q.

$$\min \quad (-3 \quad 3 \quad 0)\, x$$

$$(P) \quad \text{s.t} \quad \begin{pmatrix} 3 & 1 & 3 \\ 5 & 0 & 3 \\ 5 & 1 & 7 \\ 8 & 3 & 1 \end{pmatrix} x \begin{array}{c} \geq \\ = \\ = \\ \leq \end{array} \begin{pmatrix} 0 \\ 5 \\ 6 \\ 7 \end{pmatrix}$$

$$x_1 \leq 0 \quad x_2 \geq 0 \quad \underset{\text{不用考虑}}{\underline{x_3 \text{ free}}}$$

$$\max \quad (0 \quad 5 \quad 6 \quad 7)\, y$$

$$\text{s.t} \quad \begin{pmatrix} 3 & 5 & 5 & 8 \\ 1 & 0 & 1 & 3 \\ 3 & 3 & 7 & 1 \end{pmatrix} x \begin{array}{c} \geq \\ \leq \\ = \end{array} \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix}$$

$$y_1 \geq 0 . \quad \overset{\text{不用考虑}}{\underline{y_2 \text{ free} \quad y_3 \text{ free}}} \quad y_4 \leq 0.$$

variable in (P)	variable in (D)
$\bar{x}_1 = 0$ or $3\bar{y}_1 + 5\bar{y}_2 + 6\bar{y}_3 + 8\bar{y}_4 = -3$	$\bar{y}_1 = 0$ or $3\bar{x}_1 + 1\bar{x}_2 + 3\bar{x}_3 = 0$
$\bar{x}_2 = 0$ or $1\bar{y}_1 + 0\bar{y}_2 + 1\bar{y}_3 + 3\bar{y}_4 = 3$	$\bar{y}_2$ free 不考虑
$\bar{x}_3$ free 不考虑	$\bar{y}_3$ free 不考虑
	$\bar{y}_4 = 0$ or $8\bar{x}_1 + 3\bar{x}_2 + 1\bar{x}_3 = 7$

- Theorem : CS General Case

Let P and D be a primal-dual pair of LPs.

$\bar{x}$ be feasible for P and $\bar{y}$ be feasible for D.

Then,

$\bar{x}$ is optimal for P
$\bar{y}$ is optimal for D

$\Longleftrightarrow$

$\bar{x}$ & $\bar{y}$ satisfy CS condition

- Theorem : CS Special Case

Let P & D be as follows.

(P) max $c^T x$

s.t. $Ax \le b$
$x \ge 0$

(D) min $b^T y$

s.t. $A^T y \ge c$
$y \ge 0$.

Let $\bar{x}$ be feasible for P $\bar{y}$ be feasible for D

Then,

$\bar{x}$ optimal for P
$\bar{y}$ optimal for D

$\Longleftrightarrow$

- $\forall i = 1, \cdots, m$ we have

  $\bar{y}_i = 0$. OR $\text{row}_i(A)\bar{x} = b_i$

- $\forall j = 1, \cdots, n$ we have

  $\bar{x}_j = 0$ OR $\text{column}_j(A)\bar{y} = c_j$

constraint i in P held with equality

y_i in solution D is 0.

proof:

by weak & strong duality.

$$\boxed{\begin{array}{l}\bar{x} \text{ is optimal for } P \\ \bar{y} \text{ is optimal for } D\end{array}}$$

$$\Leftrightarrow \quad \boxed{c^T \bar{x} = b^T \bar{y}}$$

$$\Leftrightarrow \quad \boxed{\begin{array}{l} c^T \bar{x} = y^T A \bar{x} \\ y^T A \bar{x} = y^T b. \end{array}}$$

$$\Leftrightarrow \quad \boxed{c^T \bar{x} \leq \bar{y}^T A \bar{x} \leq y^T b = b^T \bar{y}}$$

$$\Leftrightarrow \quad \boxed{\begin{array}{l} (c^T - \bar{y}^T A) \bar{x} = 0 \\ y^T (A \bar{x} - b) = 0 \end{array}}$$

$$\Leftrightarrow \quad \boxed{\begin{array}{l} \sum_{j=1}^{n} \overset{\leq 0}{\underline{(c_j - \mathrm{col}_j(A)\bar{y})}}\ \overset{\geq 0}{\underline{\bar{x}_j}} = 0. \\ \sum_{i=1}^{m} \underset{\geq 0}{\underline{\bar{y}_i}}\ \underset{\leq 0}{\underline{(\mathrm{row}_i(A)\bar{x} - b_i)}} = 0 \end{array}}$$

$\Leftrightarrow$ CS holds for $\bar{x}$ & $\bar{y}$.

- Complementary Slackness Geometry

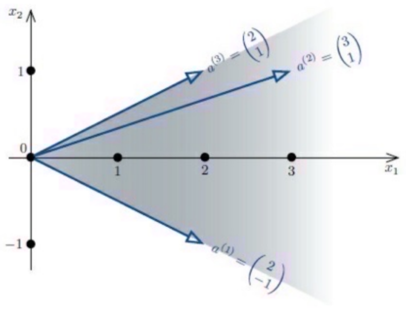
- def. cone

Given $a^{(1)}, a^{(2)} \cdots a^{(k)} \in \mathbb{R}^n$

the cone generated by $a^{(1)}, a^{(2)} \cdots a^{(k)}$ is cone

$$\{a^{(1)} \cdots a^{(k)}\} = \{\sum_{i=1}^{k} \underbrace{a^{(i)}}_{\text{direction vector.}} \cdot \lambda_i : \lambda_i \geq 0 \quad \forall i=1, \cdots, k\}$$

ep.



$$a^{(1)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad a^{(2)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad a^{(3)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \in \text{cone}\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\}$$

- def. cone of tight constraints.

Let $P := \{x : Ax \leq b\}$ be a polyhedron and let $\bar{x} \in P$

Let $J(\bar{x})$ be the row indices of A corresponding to the tight constraints of $Ax \leq b$. for $\bar{x}$. So $\forall i \in J(\bar{x})$, $\text{row}_i(A)\bar{x} = b_i$

cone of tight constraints for $\bar{x}$ is the cone C generated by rows of A correspond to tight constraints. $C = \text{cone}\{\text{row}_i(A)^T : i \in J(\bar{x})\}$.

ep.

$$\boxed{\begin{array}{l} \max \ (\tfrac{3}{2} \quad \tfrac{1}{2})x \\ \text{s.t} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} x \leq \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{matrix} \end{array}}$$

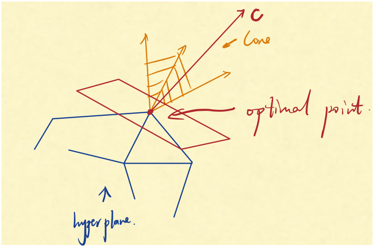
feasible sol: $\bar{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

tight constraints: ① & ② (≤ 在此处为 =)

cone of tight constraints for $\bar{x}$: $C = \text{cone}\{\underset{\textcircled{1}^T}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{\textcircled{2}^T}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}\}$.

- Theorem: Cone of tight constraints.

Let $\bar{x}$ be a feasible s.l for $\boxed{\begin{array}{ll} \max & c^T x \\ \text{s.t} & Ax \le b \end{array}}$



$\bar{x}$ is optimal $\Leftrightarrow$ c is in the cone of tight constraints for $\bar{x}$

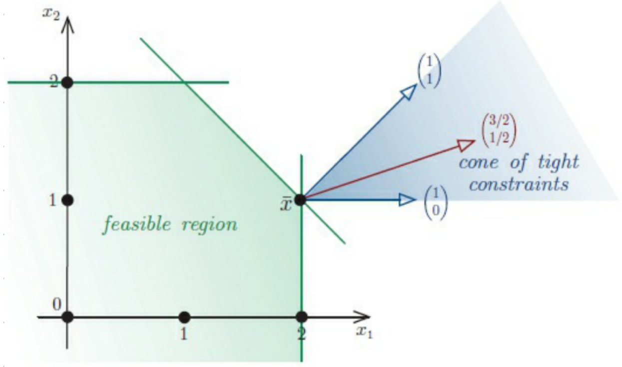
eg.

(P)
$$\boxed{\begin{array}{ll} \max & (\tfrac{3}{2}\ \ \tfrac{1}{2})x \\ \text{s.t} & \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} x \le \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \end{array}}$$

(D)
$$\boxed{\begin{array}{ll} \min & (2\ \ 3\ \ 2)\,y \\ \text{s.t} & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} y = \begin{pmatrix} \tfrac{3}{2} \\ \tfrac{1}{2} \end{pmatrix} \\ & y \ge 0 \end{array}}$$

已知 $\bar{x} = (2\ \ 1)^T$ is optimal sol

↓ cone of tight constraints for $\bar{x}$.



→ find feasible sol $\bar{y}$ for D

$$\bar{y} = \begin{pmatrix} 1 \\ \tfrac{1}{2} \\ 0 \end{pmatrix}$$

→ Verify $\bar{x}$ & $\bar{y}$ satisfy CS condition

The CS condition for $\bar{x}$ & $\bar{y}$ are

$y_1 = 0$ OR $x_1 = 2$ ✓

$y_2 = 0$ OR $x_1 + x_2 = 3$ ✓

$y_3 = 0$ ✓ OR $x_2 = 2$

$\therefore \bar{x} = (2, 1)^T$ is optimal

proof:

$$(P)\ \boxed{\begin{array}{ll} \max & c^T x \\ \text{s.t} & Ax \le b \end{array}} \quad \equiv \quad \boxed{\begin{array}{ll} \min & b^T y \\ \text{s.t} & A^T y = c \\ & y \geq 0 \end{array}}\ (D)$$

(⇒) $\bar{x}$ is optimal ⇒ $\exists \bar{y}$ optimal that follows CS

$\therefore c = A^T y = \Sigma(\bar{y}_i \, row_i (A)^T : i \in J)$ $\quad \bar{y}_i \geq 0$

(⇐) $c = \Sigma(\bar{y}_i \, row_i (A)^T : i \in J)$ $\quad \bar{y}_i \geq 0$

→ prove feasible

set 0 to $i \in J(\bar{x})$ $\Rightarrow c = A^T (\bar{y}_1, \cdots, \bar{y}_m)^T$ $\quad \bar{y} \geq 0$

→ prove CS condition

$y_i = 0$ for $i \notin J(\bar{x})$

$row_i (A) x = b$ for $i \notin J(\bar{x})$

Q

$$\max \ (1 \quad 2)\, x$$

$$\text{s.t} \quad \begin{matrix} ① \\ ② \\ ③ \end{matrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 9 \\ 7 \\ 5 \end{pmatrix} \qquad (P)$$

show $\bar{x} = (1 \quad 6)^T$ is optimal by producing a dual sol. satisfy CS condition

---

→ 写出 Dual

$$\min \ (9 \quad 7 \quad 5)\, y$$

$$\text{s.t} \quad \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} y = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad (D)$$

$$y_1 \geq 0 \quad y_2 \geq 0 \quad y_3 \geq 0$$

→ 进行验证 $x = (1, 6)$

① $2 \cdot 1 + 1 \cdot 6 = 8 < 9.$ $\bar{y}_1 = 0$

② $1 \cdot 1 + 1 \cdot 6 = 7$ $\bar{y}_2$ 不用为0

③ $-1 \cdot 1 + 1 \cdot 6 = 5$ $\bar{y}_3$ 不用为0

原因: CS condition

x.y are optimal ⟺.

$\langle A_i, x \rangle = b_i$ or $y_i = 0$

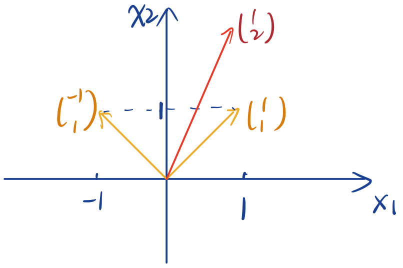
→ 解 $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \bar{y}_2 + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \bar{y}_3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \bar{y}_2 .\ \bar{y}_3 \geq 0$$

$$\bar{y}_2 = \frac{3}{2} \qquad \bar{y}_3 = \frac{1}{2}$$

∴ 解为 $(0, \frac{3}{2}, \frac{1}{2})$

结合图像理解上题 eg. cone $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$



$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \text{cone} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

$$\text{since } \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{3}{2} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \frac{1}{2}$$

— Ferkas' Lemma.

- Ferka's Lemma.

Given $A \in R^{m\times n}$. $b \in R^m$. exactly one ↓ is true:

i) $\exists x$. s.t. $Ax=b$ $x \geq 0$

ii) $\exists y$. s.t $A^T y \geq 0$. $b^T y < 0$.

proof:

法 1: Simplex method

P

$$\begin{aligned} &\max \quad 0^T x \\ &\text{s.t} \quad Ax=b \\ &\qquad\quad x \geq 0 \end{aligned}$$

terminate with exactly 1 of outcomes:

① ∃ feasible x.

s.t $Ax=b$ $x \geq 0$

② ∃ certificate of infeasibility: y.

s.t. $A^T y \geq 0$. $b^T y < 0$

法二: Primal - Dual

$$\begin{aligned} &\min \quad b^T y \\ &\text{s.t} \quad A^T y \geq 0 \end{aligned}$$

D

D is feasible.

since $y=0$ is a feasible sol for D.

exactly 1 case happen:

① (D) optimal ⇒ (P) optimal

$\exists y$ s.t $A^T y \geq 0$. $b^T y < 0$

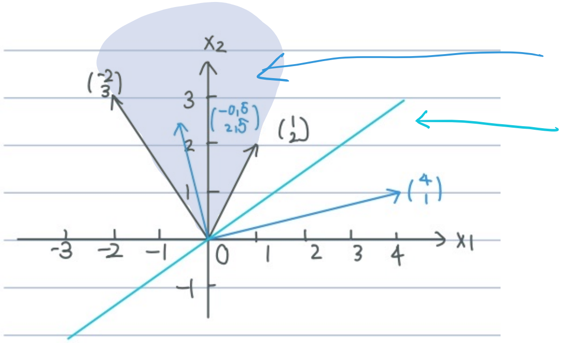
② (D) unbounded ⇒ (P) feasible

$\exists x$. s.t $Ax=b$ $x \geq 0$

还待证明 i) 和 ii) 无法同时发生

Q. Consider cone $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-2\\3\end{pmatrix}\right\} = \left\{\begin{pmatrix}1\\2\end{pmatrix}\lambda_1 + \begin{pmatrix}-2\\3\end{pmatrix}\lambda_2 : \lambda_1, \lambda_2 \geq 0\right\}$

$$= \left\{ b : \begin{pmatrix}1 & -2\\2 & 3\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = b.\ \lambda = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \geq 0 \right\}$$



the defined cone

separating hyperplane

$\{x \in \mathbb{R}^2 : (-1 \quad 1)\, x = 0\}$

→ How to show $b = \begin{pmatrix}-0.5\\2.5\end{pmatrix}$ lies in cone?

provide non-neg coefficient for λ.

$$\begin{pmatrix}-0.5\\2.5\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}\cdot \underbrace{0.5}_{\lambda_1} + \begin{pmatrix}-2\\3\end{pmatrix}\cdot \underbrace{0.5}_{\lambda_2}.$$

→ How to show $b = \begin{pmatrix}4\\1\end{pmatrix}$ doesn't lie in cone?

provide a separating hyperplane.

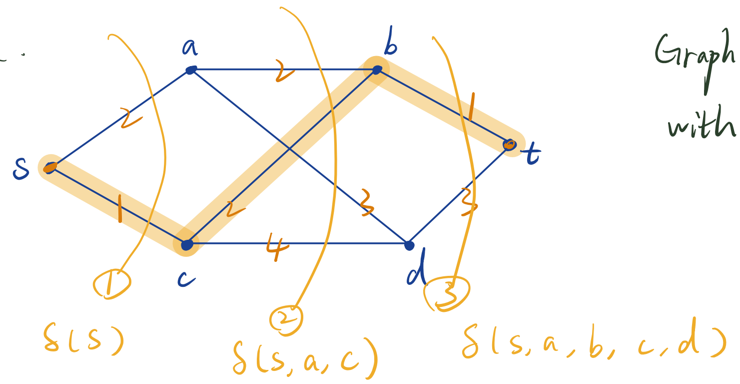
consider $\underbrace{(-1 \quad 1)}_{y^T} x \leq 0$

$(-1 \quad 1)\begin{pmatrix}1\\2\end{pmatrix} \geq 0 \qquad (-1 \quad 1)\begin{pmatrix}-2\\5\end{pmatrix} \geq 0$

but $(-1 \quad 1)\begin{pmatrix}4\\1\end{pmatrix} < 0$

4. Shortest Path Problem

Q.



Graph: $G=(V,E)$

with costs $c_e, e\in E$

$c_e \geqslant 0$

$e \in E$.

- consider an st-path P: sc, cb, bt.

  the length of (P) is denoted by $c(P) = 1+2+1 = 4$

(P)

$$\begin{aligned} \min \ & \sum_{e\in E} c_e \cdot x_e \\ \text{s.t.} \ & \sum_{e\in\delta(u)} x_e \geqslant 1 \quad \forall U \subseteq V. \ s\in U \ \ t\notin U \\ & x \geqslant 0 \end{aligned}$$

(D)

$$\begin{aligned} \max \ & \sum (y_u : \delta(u) \text{ is st-cut}) \\ \text{s.t} \ & \sum (y_u : \delta(u) \text{ is st-cut}, e\in\delta(u)) \leqslant c_e. \quad \overset{e\in E}{} \\ & y \geqslant 0 \end{aligned}$$

→ 理解 y_u

- the cut constraint for $U = \{\underline{s, a, c}\}$ is: ②

$$\underline{x_{cd} + x_{ad} + x_{cb} + x_{ab}} \geqslant \underline{1}$$

↑ 被②切到的所有线

x_{cd}: cd 这条边是否要被经过

若从 s 到 t 必经经过 cut ②.

∴ cd. ad. cb. ab 中至少有一条线被经过

The optimal val of P is at most the length of a shortest path

- constraint in (D) corresponding to edge "ad" is: ← e 的一个例子

$$\underline{y_{\{s,a\}} + y_{\{s,a,c\}} + y_{\{s,a,b\}} + y_{\{s,a,b,c\}} + y_{\{s,d\}} + y_{\{s,a,c\}} + y_{\{s,a,b\}} + y_{\{s,b,c,d\}}} \leqslant \underline{3}$$

所有包含 ad 边的 st-cut

⇒ y_u: s-t cut (u) 的最大长度

⇒ $y_u \leqslant$ 所有包含 u 的 edge

ad 的长度 ⟶ C_e ↑

若从 a 到 d 最多只能增加 3 个单位长度

- Show P is a shortest st-path.

consider LP relaxation.

Show there are no st-paths of length ≤ 4.

→ length of shortest path is at least the optimal val of (P)

$\therefore$ at least the optimal val of (D) by strong duality

→ We can provide a feasible sol y for (D^*) with $val_{D^*}(y) = 4$

consider $y_{\{s\}} = 1$, $y_{\{s,a,c\}} = 2$, $y_{\{s,a,b,c,d\}} = 1$, $y_u = 0$ (otherwise)

cut ① 所包含的点 ② ③

在每条 st-cut 中取最短的，且能互相连接的边 的长度

↓

简单来说：最短路径.

$$\begin{cases} \text{其它的 edge} & 0 < C_e \\ \text{包含任意一条的 edge} & \begin{cases} y_{\{s\}} = 1 & \leq sa \,/\, sc \\ y_{\{s,a,c\}} = 2 & \leq ab \,/\, bc \,/\, ad \,/\, cd \\ y_{\{s,a,b,c,d\}} = 1 & \leq bt \,/\, dt \end{cases} \end{cases}$$

↓ ↙

所选中的 edge 是所属 cut 中最短的路径.

家人们谁懂啊. 这TM是个特殊情况.

TMD不一定每个 cut 的最短边都能互相相连的 F**k.

无语

Primal-Dual Algorithm

- def. arc & directed st-path

arc : $\overrightarrow{uv}$ （尾→头）

directed st-path : sequence of arcs.

- width (y_w)

w : $\delta(w)$ is an st-cut $y_u \geq 0$.

y_w : w 所在 cut 的值.

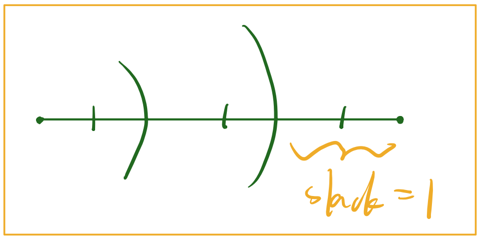
y is feasible width ⟺ 同时满足 $\begin{cases} y \geq 0 \\ 0 \leq \text{slack } y(e) = c_e - \Sigma(y_w : \delta(w) \text{ is st-cut}) \end{cases}$

$\forall e \in E$

The dual LP.

$$\boxed{\begin{array}{ll} \max \Sigma(y_w : \delta(w) \text{ is st-cut}) & \\ \text{s.t. } \Sigma(y_w : \delta(w) \text{ is st-cut}) \leq c_e & \text{for } e \in E \\ \quad y \geq 0 \rightarrow \text{width.} & \end{array}}$$

↓ feasible width



- slack of edge e

剩下的路程.

length of e − total width of all st-cuts using e

$\text{slack}_y(e) = c_e - \Sigma(y_u : \delta(u) \text{ is an st-cut containing } e)$

- equality edge & active cut

Let $\bar{y}$ be a feasible sol to D.

$\text{slack}_y(e) = 0$ ⇒ edge e is an equality edge

$\bar{y}_u > 0$ ⇒ cut $\delta(u)$ is active for $\bar{y}$.

被激活的

- Proposition

Let $\bar{y}$ be feasible widths. P be an st-path

Then P is a shortest st-path if both statement hold.

i) all edges in P are equality edges for $\bar{y}$ → tight constraints

ii) all active cut for $\bar{y}$ containing exactly 1 edge of P.

Proof

Feasible Condition: For every edge $e \in E$, the total with of all st-cut contain e doesn't exceed length of e

total width of all st-cut

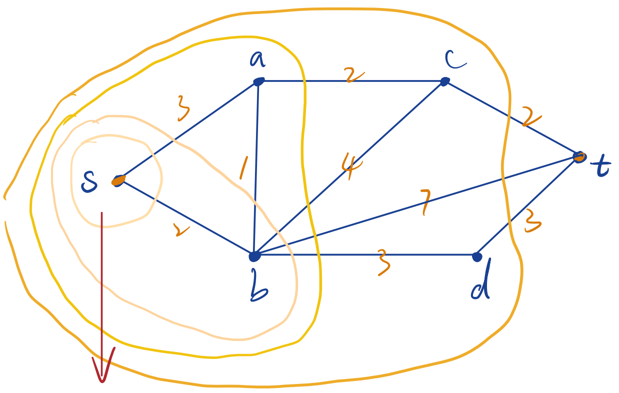
$= \Sigma (y_u : \delta(u)$ is st-cut$)$

$= \sum_{e \in P} (\Sigma (y_u : \delta(u)$ is active cut$))$ by (2) exactly 1

$= \sum_{e \in P} c_e$ by (1) $\text{slack}_{\bar{y}}(e) = 0 \Rightarrow c_e = \Sigma y_u$

= length of P

Thus, by Strong Duality, P is the shortest path



相当于圆从 s 开始不停往外扩张

iteration 1 $\quad y = 0 \quad U = \{s\}$

$y_{\{s\}} \leftarrow 2 \quad U \leftarrow \{s, b\}$

iteration 2 $\quad y_{\{s,b\}} \leftarrow 1 \quad U \leftarrow \{s, b, a\}$

iteration 3 $\quad y_{\{s,a,b\}} \leftarrow 2 \quad U \leftarrow \{s, b, a, c, d\}$

iteration 4 $\quad y_{\{s,b,a,c,d\}} \leftarrow 2 \quad U \leftarrow V.$

感觉是解释 (D) 中的 y 是干嘛的

$$(P) \quad \begin{aligned} \min \quad & \sum_{e \in E} c_e^T \cdot x_e \\ \text{s.t} \quad & \sum_{e \in \delta(u)} x_e \geq 1 \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & \sum (x_e : e \in E) \\ \text{s.t.} \quad & \sum (x_e : e \in \delta(S)) \geq 1, \ \ (\delta(S) \text{ is } s,t\text{-cut}) \\ & x \geq \mathbb{0} \end{aligned}$$

$$(D) \quad \begin{aligned} \max \quad & \mathbb{1}^T y \\ \text{s.t} \quad & \sum_{\delta(u)} y_u \leq c_e \quad \forall e \in E \\ & y \geq 0 \end{aligned}$$

s.t cut

$$\begin{aligned} \max \quad & \sum (y_S : \delta(S) \ s,t\text{-cut}\ ) \\ \text{s.t.} \quad & \sum (y_S : e \in \delta(S)) \leq c_e \ \ (e \in E) \\ & y \geq \mathbb{0} \end{aligned}$$

- Alogorithm

输入： Graph $G=(V,E)$， 每个 $e\in E$ 的值为 $c_e\geq 0$.

输出： shortest st-path.

Step 1. (Initialization)

$y_w \leftarrow 0$ for st-cut $\delta(w)$ 初始时已行驶距离为0

$u \leftarrow \{s\}$: set of vertices that can be reached 起点为u.

Step 2. If $t \notin U$

→ $slack_y(ab)$: $\min\{c_{ab} - \Sigma(y_w : ab \text{ is the st-cut } \delta(w))\}$

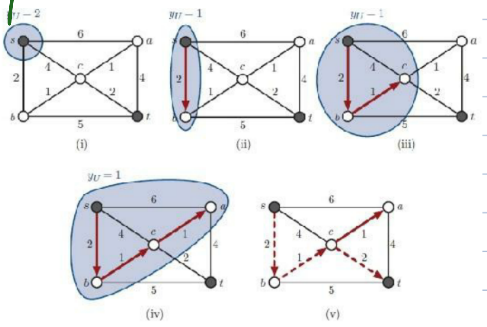
→ $y_u \leftarrow slack(ab)$

→ $U \leftarrow U \cup \{b\}$

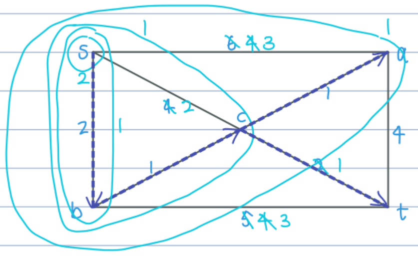
→ change ab into arc $\vec{ab}$ 循环，直到到达t停止

Step 3 : return st-path P that uses the constructed arcs $\{\vec{ab_1}, \vec{ab_2}, \dots, \vec{ab_n}\}$

ep

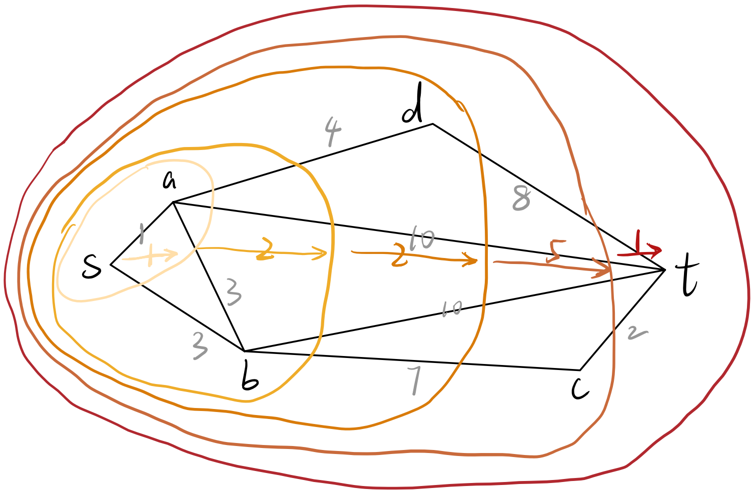


(i) (ii) (iii) (iv) (v)



I0	$y=0$	$u=\emptyset$	$P=\{\vec{sb}, \vec{bc}, \vec{ct}\}$
I1	$y\{s\}\leftarrow 2$	$u\leftarrow \{s,b\}$	$c(P)=2+1+2=5$
I2	$y\{s,b\}\leftarrow 1$	$u\leftarrow \{s,b,c\}$	$\Sigma y_w=2+1+1=5$
I3	$y\{s,b,c\}\leftarrow 1$	$u\leftarrow \{s,b,c,a\}$	so $value_P(\bar{x}) = value_D(\bar{y})$
I4	$y\{s,b,c,a\}\leftarrow 1$	$u\leftarrow \{s,b,c,a,t\}$	$slack_{\bar{y}}(sb) = slack_{\bar{y}}(bc) = slack_{\bar{y}}(ct) = 0$

Q.



增加 in slackness

在圈里的点

			sa	sb	ab	at	ad	dt	ct	bc
iteration 0	$y \leftarrow 0$	$u \rightarrow \{s\}$	1	3 -2	3	10	4	8	2	7
iteration 1	$y_{\{s\}} \leftarrow 1$	$\{s,a\}$	0	2 -2	3 -2	10 -2	4 -2	8	2	7
iteration 2	$y_{\{s,a\}} \leftarrow 2$	$\{s,a,b\}$	0	0	1	8 -2	2 -2	8	2	7 -2
iteration 3	$y_{\{s,a,b\}} \leftarrow 2$	$\{s,a,b,d\}$	0	0	1	6 -5	0	8 -5	2	5 -5
iteration 4	$y_{\{s,a,b,d\}} \leftarrow 5$	$\{s,a,b,c,d\}$	0	0	1	1 -1	0	3 -1	2 -1	0
iteration 5	$y_{\{s,a,b,c,d\}} \leftarrow 1$	all vertices	0	0	1	0	0	2	1	0

↑ Stop! Since $t \in U$

output: sa, at ← use only the constucted arcs

observation:

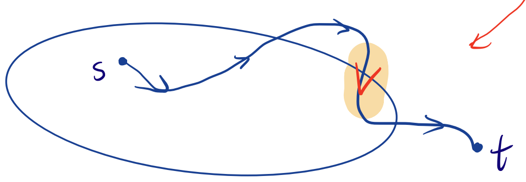
(i) y are feasible width throughout the algorithm

(ii) all created arcs are equality edges.

(iii) if W is active: $y_W > 0$.

there is no one of the form $\overrightarrow{uv}$, $u \in W$, $v \in W$.

出去了不可能再回来



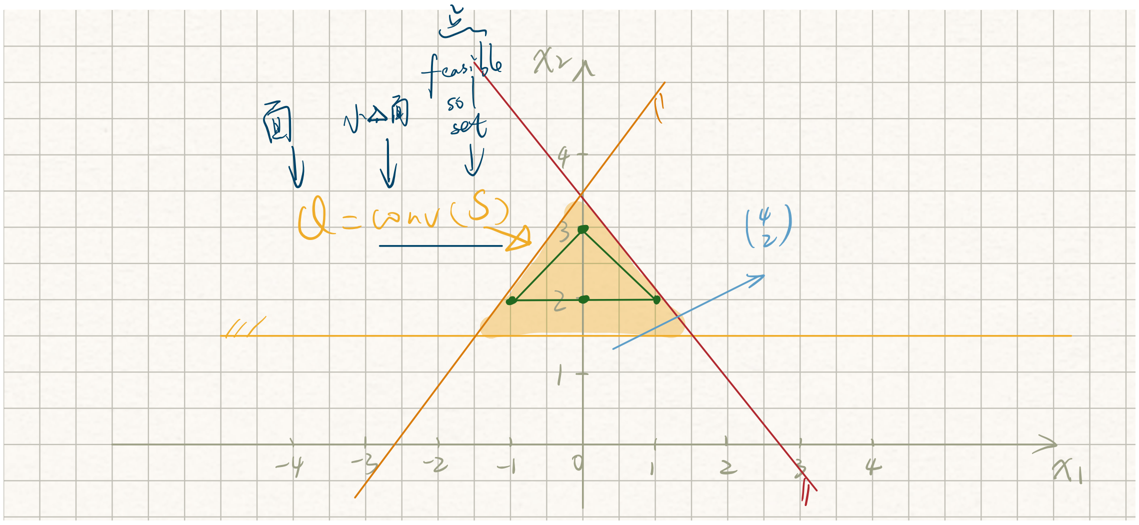
5. Integer Programming

Q.

$$\max \quad (4 \quad 2)\, x$$

$$\text{s.t} \quad \begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{pmatrix} 8 & 6 \\ -8 & 6 \\ 0 & -1 \end{pmatrix} x \le \begin{pmatrix} 21 \\ 21 \\ -\frac{3}{2} \end{pmatrix}$$

x integer.



The feasible sol form a set : $S = \{\binom{-1}{2}, \binom{1}{2}, \binom{0}{2}, \binom{0}{3}\}$

将4个值代入计算，或画出 $y = 4x_1 + 2x_2$ (向上平移)

得出 $\binom{1}{2}$ optimal

- Suppose we can find a convex set. (polyhedron) Q. s.t $Q \supseteq S$ and all extreme points of Q belong to S. after that we maximize the objective function over Q.
- $Q = \mathrm{conv}(S) = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} x \le \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} \right\}$ ← 三角形的3条边

and so we can solve $(4 \quad 2)\, x$

- def. convex hull

Let S be a set of points in $\mathbb{R}^n$.

The convex hull of S, denoted by $\text{conv}(S)$, is the smallest convex set containing S.

- Fundamental Theorem of IP.

Consider the following IP. $\max\{c^T x : Ax \le b,\ x \text{ integer}\}$

Let S be the set of all feasible solutions of this IP.

Then the convex hull $\text{conv}(S)$ is a polyhedron.

$Q = \{x \in \mathbb{R}^n : \tilde{A}x \le b\}$

Consider LP. $\max\{c^T x : \tilde{A}x \le b\}$.

接下来3条得出来的值一样:

1) LP infeasible. $\Longleftrightarrow$ IP infeasible

2) LP unbounded. $\Longleftrightarrow$ IP unbounded

3) LP optimal. $\Longleftrightarrow$ IP optimal sol.

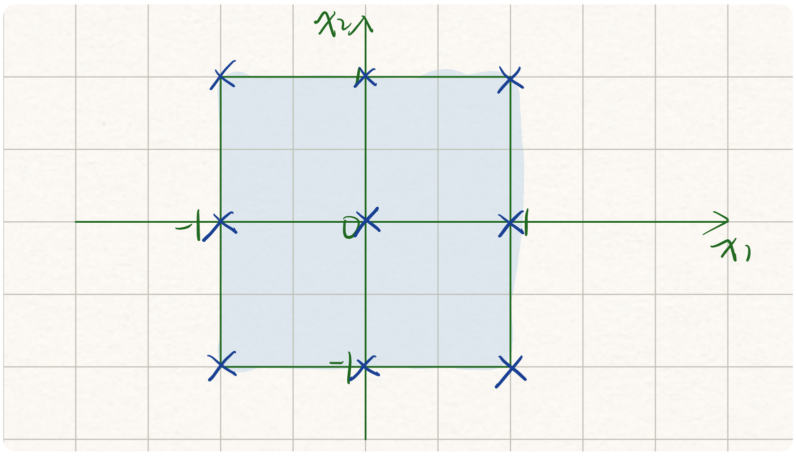
4) every optimal sol for IP is optimal for LP.

every optimal sol for LP is optimal for IP

→ is an extreme point of Q

Q.

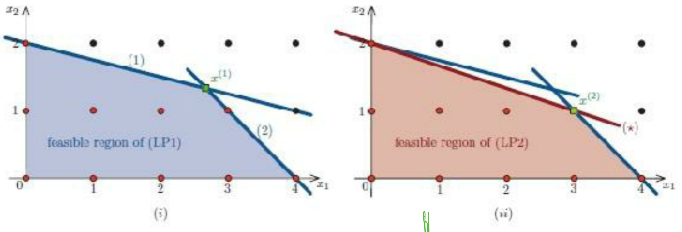
$$\boxed{\begin{array}{ll} \max & (1 \quad 0)x \\ \text{s.t} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} x \in \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ & x \in \mathbb{Z} \end{array}}$$



$$S = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$$

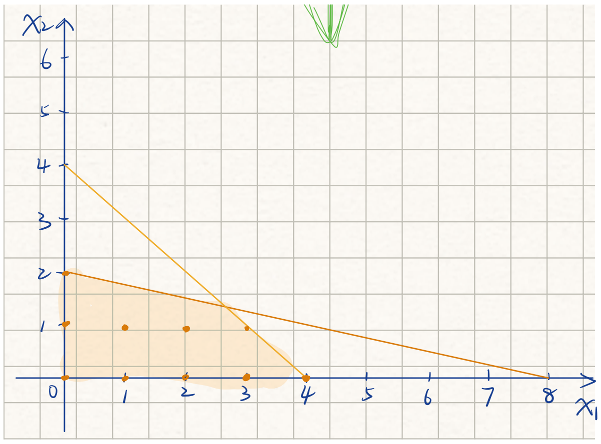
↖ 图中(X)标记的9个点

$\begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$ is optimal for LP, but not feasible for IP.



→ Cutting Planes

$$\max \ (2 \quad 5)\ x$$
$$\text{s.t} \ \begin{matrix}①\\②\end{matrix}\begin{pmatrix}1 & 4\\ 1 & 1\end{pmatrix} x \le \begin{pmatrix}8\\4\end{pmatrix}$$
$$x \geqslant 0 \qquad x \in \mathbb{Z}.$$



→ Step 1

$$\max \ (2 \quad 5 \quad 0 \quad 0)\ x$$
$$\text{s.t} \ \begin{pmatrix}1 & 4 & 1 & 0\\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}8\\4\end{pmatrix}$$
$$x \geqslant 0 \quad \boxed{x \in \mathbb{Z}} \longleftrightarrow$$

LP relaxation : remove it.

- obtain a final dictionary:

→ $B = \{3, 4\}$.

$z = 2x_1 + 5x_2$.

8 $x_3 = 8 - x_1 - 4x_2$

4 $x_4 = 4 - x_1 - x_2$.

$\downarrow$

min 4 $\quad x_4$ leave. x_1 enter.

→ $B = \{1, 3\}$

$z = 8 + \underline{3x_2} - 2x_4$

4 $x_1 = 4 - x_2 - x_4$.

$\frac{4}{3}$ $\underline{x_3} = 4 - 3x_2 - x_4$

$\downarrow$

min $\frac{4}{3}$ $\quad x_3$ leave x_2 entre.

→ $B = \{1, 2\}$

$z = 12 - x_3 - x_4$

$x_1 = \frac{8}{3} + \frac{1}{3}x_3 - \frac{2}{2}x_4$

$x_2 = \frac{4}{3} - \frac{1}{3}x_3 - \frac{1}{3}x_4$

$$\boxed{\begin{aligned} z &= 12 - x_3 - x_4 \\ x_1 &= \frac{8}{3} + \frac{1}{3}x_3 - \frac{4}{3}x_4 \\ x_2 &= \frac{4}{3} - \frac{1}{3}x_3 + \frac{1}{3}x_4 \end{aligned}}$$

使用Phase I iteration
找到1个解.

for $B = \{1, 2\}$ with basic solution $\bar{x} = (\frac{8}{3}, \frac{4}{3}, 0, 0)^T$

- find $\alpha_1 x_1 + \alpha_2 x_2 \le \beta$.

s.t 1) $\alpha_1 x_1 + \alpha_2 x_2 \le \beta$, holds for every feasible sol of (P)

2) $\alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 > \beta$ $\quad \bar{x}$ does NOT satisfy (*)

Use factorial coordinate of $\bar{x}$ to construct a cutting plane.

Consider $\bar{x}_1 = \frac{8}{3}$, the corresponding constant in the dictionary.

$x - \frac{1}{3}x_3 + \frac{4}{3}x_4 = \frac{8}{3}$ $\quad$ holds for every x feasible for (IP*)

$\lfloor 1 \rfloor x_1 + \lfloor -\frac{1}{3} \rfloor x_3 + \lfloor \frac{4}{3} \rfloor x_4 \le \frac{8}{3}$ holds for every x feasible for (IP^*)

$$
\begin{aligned}
1 \cdot x_1 - 1\, x_3 + 1\, x_4 &\le \frac{8}{3} \\
&\le \left\lfloor \frac{8}{3} \right\rfloor \quad (\because x \in \mathbb{Z} \ \therefore x_1 - x_3 + x_4 \in \mathbb{Z}) \\
&\le 2
\end{aligned}
$$

← (如果题目中不要 justification，上述步可省)

- 回到 original (IP)

$x_1 - (8x_1 - 4x_2) + (4 - x_1 - x_2) = x_1 + 3x_2 - 4 \le 2.$

$\therefore x_1 + 3x_2 \le 6$ → 可得 $(\frac{8}{3}, \frac{4}{3})$ 不满足条件

How x_3 & x_4 were added in IP.

$x_3 = 8 - x_1 - 4x_2$

$x_4 = 4 - x_1 - x_2$ both are $\mathbb{Z}$ when $x_1, x_2 \in \mathbb{Z}$

→ Step 2

consider

$$
\boxed{
\begin{aligned}
&\max \ (2 \quad 5)\, x \\
&\text{s.t} \begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} \\
&x \ge 0 . \quad x \in \mathbb{Z}.
\end{aligned}
}
$$

Solve the LP relaxation of

$$
\boxed{
\begin{aligned}
&\max \ (2 \quad 5 \quad 0 \quad 0 \quad 0)\, x \\
&\text{s.t} \begin{pmatrix} 1 & 4 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} \\
&x \ge 0 \quad x \in \mathbb{Z}
\end{aligned}
}
$$

We obtain the last directory for $B = \{1, 2, 3\}$.

$$z = 11 - \frac{1}{2} x_4 - \frac{3}{2} x_5$$

$$
\begin{aligned}
x_1 &= 3 - \frac{3}{2} x_4 + \frac{1}{2} x_5 \\
x_2 &= 1 + \frac{1}{2} x_4 - \frac{1}{2} x_5 \\
x_3 &= 1 - \frac{1}{2} x_4 + \frac{3}{2} x_5
\end{aligned}
$$

The optimal basic solution is $\bar{x} = (3, 1, \underline{1, 0,} 0)^T$

$\therefore \binom{3}{1}$ is optimal for the original IP.

- def. cutting plane

Given a IP and a feasible solution $\bar{x}$ for it LP relaxation.

on an inequality $a^T x \le \beta$ (*) is a cutting plane for $\bar{x}$.

if 1) (*) is valid for all feasible sol of the IP. feasible region 内

2) $\bar{x}$ does not satisfy (*) feasible region 外

Q. Construct all possible cutting planes for $\bar{x} = (1.3, 3.3, 0, 0)^T$ which are coming from rows in the LP.

$$
\boxed{
\begin{array}{ll}
\max & -0.58x_3 - 0.76x_4 \\
\text{s.t} & -0.2x_3 + 0.1x_4 = 1.3 \\
 & x_2 + 0.8x_3 + 0.1x_4 = 3.3 \\
 & x \ge 0. \quad x \in \mathbb{Z}.
\end{array}
}
$$

- from 1st row, for every feasible x we have

$$x_1 - 0.2x_3 + 0.1x_4 = 1.3$$

$x \ge 0$

$$\lfloor 1 \rfloor x_1 + \lfloor -0.2 \rfloor x_3 + \lfloor 0.1 \rfloor x_4 \le 1.3$$

$$1 \cdot x_1 - 1 \cdot x_3 \le 1.3.$$

$\because x \in \mathbb{Z} \quad \therefore x_1 - x_3 \in \mathbb{Z}$

$$1 \cdot x_1 - 1 \cdot x_3 \le 1$$

The proof above show that $1x_1 - 1x_3 \le 1$ holds for all feasible IP.

$1 \cdot \bar{x}_1 - 1 \cdot \bar{x}_3 = 1.3 > 1$

$\therefore$ $x_1 - x_3 \le 1$ is a cutting plane for $\bar{x}$

- from 2nd row, for every feasible x we have

$$x_1 - 0.8x_3 + 0.1x_4 = 3.3$$

$x \ge 0$

$$\lfloor 1 \rfloor x_1 + \lfloor -0.8 \rfloor x_3 + \lfloor 0.1 \rfloor x_4 \le 3.3$$

$$x_2 \le 3.3$$

$x \in \mathbb{Z}$

$$x_2 \le 3.$$

Q. Is $\boxed{6x_3+7x_4 \geq 21}$ a valid cutting plane for $\bar{x} = (1.3, 3.3, 0, 0)^T$ in the previous example? $\rightarrow$ $6x_3+7x_4 \geq 21$

→ Let us check both properties of cutting planes

2) $6 \cdot 0 + 7 \cdot 0 = 0 < 21$

$\therefore$ (*) is not satisfied by $\bar{x}$

1) Let us show that (*) is satisfied by all feasible x for the IP.

→ Let us assume there is a feasible x. for the IP s.t $6x_3+7x_4 < 21$

We have 3 possible cases:

case 1: $x_4 = 2 \Rightarrow x_3 = 0$ or $x_3 = 1$

$(x_1 - 0.2x_3) - \lfloor (x_1 - 0.2x_3) \rfloor \in \{0, 0.8\}$

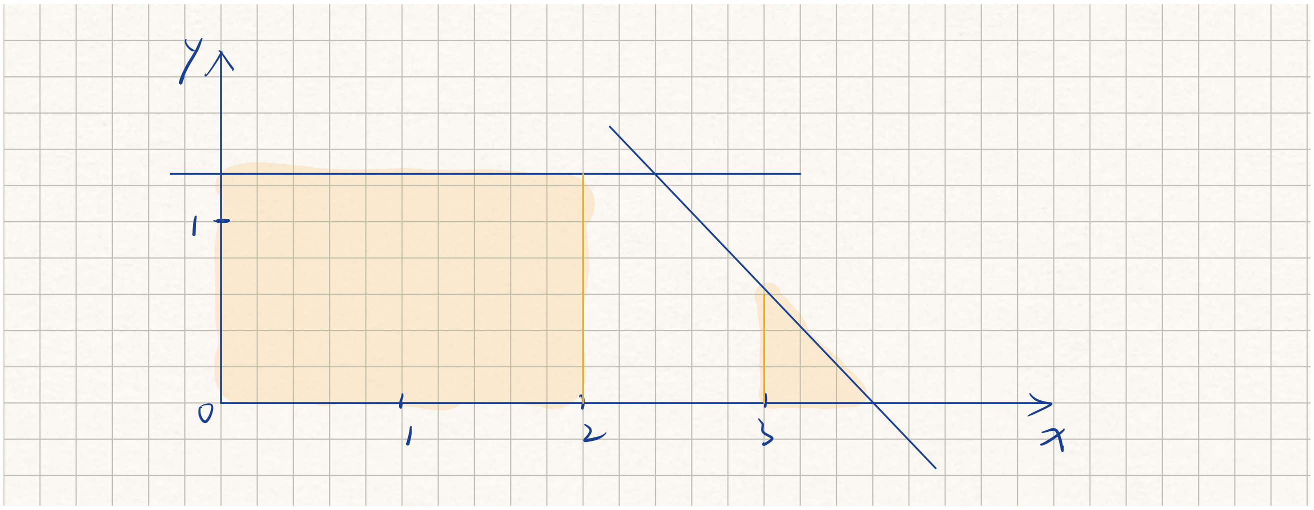
but $(1.3 - 0.1x_4) - \lfloor 1.3 - 0.1x_4 \rfloor = 0.1$

So the 1st guess contradiction

case 2: $x_4 = 1$

case 3: $x_4 = 0$

自己补齐喽


y
1
0
1
2
3
x

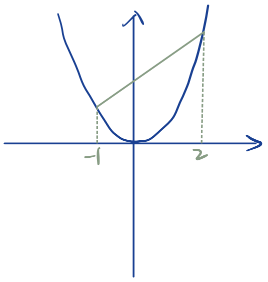
7. Non-linear Optimization Convexity

- def. convex function

A function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if every $u, v \in \mathbb{R}^n$ and for every $\lambda \in [0, 1]$ we have $f(\lambda u + (1-\lambda)\theta) \le \lambda f(u) + (1-\lambda) f(u)$

ep.

a) $f(x) = x^2$

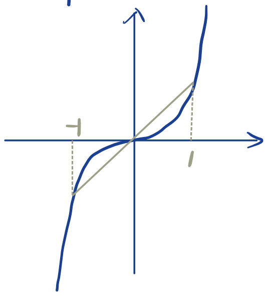


check def property for $u=-1$ $\theta=2$. $\lambda=\frac{4}{3}$

$$f(\underbrace{\lambda u+(1-\lambda)\theta}_{1}) = 1 \le \lambda f(u) + (1-\lambda) f(\theta) = \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot 4 = 3.$$

IS convex function

b) $f(x) = x^3$



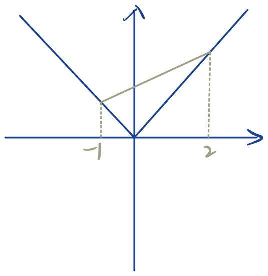
Not convex function

Take $u=0$ $\theta=-1$ $\lambda=\frac{1}{2}$

$$f(\underbrace{\lambda u+(1-\lambda)\theta}_{-\frac{1}{2}}) = -\frac{1}{8}$$

$$\lambda \underbrace{f(u)}_{0} + \underbrace{(1-\lambda)}_{\frac{1}{2}} \underbrace{f(\theta)}_{-1} = -\frac{1}{2}$$

c) $f(x) = |x|$



Convex function.

证明 consider $u \in \mathbb{R}$. $\theta \in \mathbb{R}$. $\lambda \in [0,1]$.

we need to show $|\lambda u + (1-\lambda)\theta| \le \lambda |u| + (1-\lambda)|\theta|$

$$|\lambda u + (1-\lambda)\theta| \le \underbrace{|\lambda u|}_{\geq 0} + \underbrace{|(1-\lambda)\theta|}_{\geq 0}$$

$$= \lambda |u| + (1-\lambda)|\theta|$$

- A non-linear program is an optimization problem of the following type

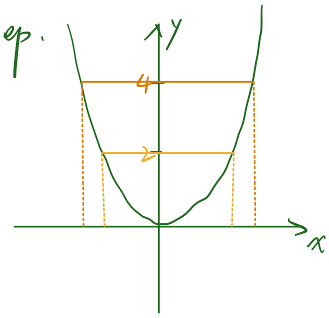
$$\boxed{\begin{array}{ll} \min & f(x) \\ \text{s.t} & g_1(x) \le 0 \\ & g_2(x) \le 0 \\ & \vdots \\ & g_m(x) \le 0 \end{array}} \qquad (NLP)$$

→ show that if all $g_1(x), \cdots, g_m(x)$ are convex function, then the feasible region of an (NLP) is a convex set (接下来都在 show 这个)

- def. level set

Let $g: \mathbb{R}^n \to \mathbb{R}$ be a function. $g \in \mathbb{R}$. then the set $\underline{\{x \in \mathbb{R}^n : g(x) \le \beta\}}$ is called level set of the function g.

ep.



$[-\sqrt{2}, \sqrt{2}]$ is a level set of $g(x) = x^2$

$[-2, 2]$ is also a level set of $g(x) = x^2$

- Proposition

Every level set of convex function is a convex set.

proof: Let g be a convex function, $g: \mathbb{R}^n \to \mathbb{R}$. $\beta \in \mathbb{R}$

To show that the level set $S = \{x \in \mathbb{R}^n. g(x) \le \beta\}$ is convex.

Let u & ϑ be in S. $\lambda \in [0, 1]$.

We need to show that $\lambda u + (1-\lambda)\vartheta \in S$. $\Rightarrow g(\lambda u + (1-\lambda)\vartheta) \le \beta$

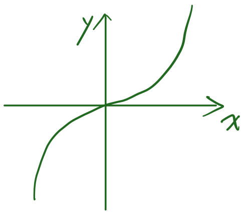
By convexity of the function g, we have

$$g(\lambda u + (1-\lambda)\vartheta) \le \underset{\ge 0}{\underline{\lambda}}\, \underset{\le \beta}{\underline{g(u)}} + \underset{\ge a}{\underline{(1-\lambda)}}\, \underset{\le \beta.}{\underline{g(\vartheta)}}$$

$$\le \lambda\beta + (1-\lambda)\beta = \beta. \quad \blacksquare$$

* There are function not convex. but every level set is a convex set

ep.



for $f(x)=x^3$, every level set has the form $[-\infty, \sqrt[3]{\beta}]$ which is a convex set

- def. convex NLP

An NLP $\boxed{\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_1(x) \le 0 \\ & \vdots \\ & g_n(x) \le 0 \end{array}}$ is a convex NLP, if all $g_1 \cdots g_n$. f. are convex function.

- Proposition

The feasible region of a convex NLP is a convex set.

proof: Since $g_1, \cdots, g_m$ are convex functions.

the level sets $\{x : g_i(x) \le 0\}$ are convex sets for $i=1, \cdots, m$.

the intersection of convex sets is a convex set.

Thus, the feasible region $\bigcap_{i=1}^{m} \{x \in \mathbb{R}^n : g_i(x) \le 0\}$ is also a convex set

- def. supporting halfspace

Consider a convex set $C \subseteq \mathbb{R}^n$. Let $\bar{x} \in C$.

The half space $F = \{x \in \mathbb{R}^n : s^T x \le \beta\}$ ($s \in \mathbb{R}^n$. $\beta \in \mathbb{R}$) is a supporting halfspace of C at $\bar{x}$ if the following condition hold:

1) $C \subseteq F$

2) $s^T \bar{x} = \beta$.

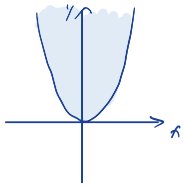
i.e. $\bar{x}$ is the only hyperplane that defines the boundary of F.

– Non-linear programming epigraphs.

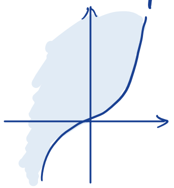
$$\begin{tikzcd}
\boxed{g \text{ is a convex function}} \arrow[r, bend left, "\checkmark"] & \boxed{\text{every level set of } g \text{ is a convex set}} \arrow[l, bend left, "\times"]
\end{tikzcd}$$

ep. $f(x)=x^3$

? Is there a stronger relationship between convex functions and convex set ?



$f(x)=x^2$



$f(x)=x^3$

– def. epigraph

Given a function $f: \mathbb{R}^n \to \mathbb{R}$. the epigraph of f is the set

$$epi(f) = \{ \binom{\alpha}{x} \in \mathbb{R} \times \mathbb{R}^n : f(x) \le \underline{\alpha} \}$$ α是f(x)所有上方的点

– proposition.

a function $f: \mathbb{R}^n \to \mathbb{R}$ is a convex function $\Leftrightarrow$ $epi(f)$ is a convex set

proof:

(⇒) Assume f is convex. $\binom{\alpha_1}{u} \in epi(f)$, $\binom{\alpha_2}{u} \in epi(f)$, $\lambda \in [0,1]$

$f(u) \le \alpha_1 \quad f(v) \le \alpha_2$

$\therefore f(\lambda u + (1-\lambda) v) \le \lambda f(u) + (1-\lambda) f(v) \le \lambda \alpha_1 + (1-\lambda)\alpha_2$.

$\therefore \binom{\lambda \alpha_1 + (1-\lambda)\alpha_2}{\lambda u + (1-\lambda) v} \in epi(f)$. $\Rightarrow$. a convex set.

(⇐) Assume $epi(f)$ is a convex set.

Then $\binom{f(u)}{u} \in epi(f)$ $\binom{f(v)}{v} \in epi(f)$ $\lambda \in [0,1]$.

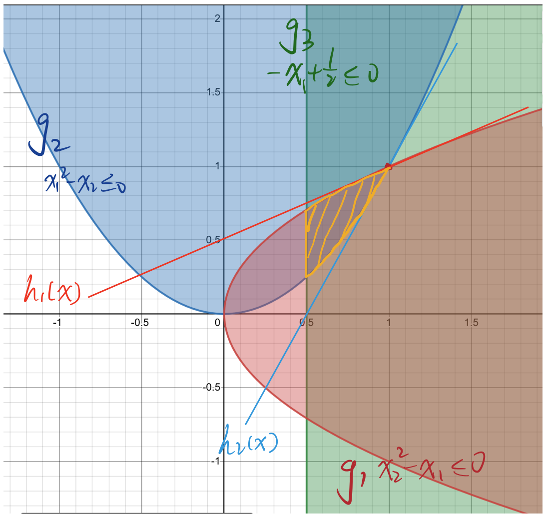
$\Rightarrow \binom{\lambda f(u) + (1-\lambda) f(v)}{\lambda u + (1-\lambda) f(v)} \in epi(f)$

$\therefore f(\lambda u + (1-\lambda) v) \le \lambda f(u) + (1-\lambda) \binom{f(v)}{v} \in epi(f)$

Non-linear programming: Karush-Kuhn-Tucker theorem

Q.

$$\boxed{\begin{aligned} \min \quad & c^T x = c_1 x_1 + c_2 x_2 \\ \text{s.t} \quad & g_1(x) \le 0 \\ & g_2(x) \le 0 \\ & g_3(x) \le 0 \end{aligned}} \quad (NLP)$$



where $g_1 = x_2^2 - x_1$

$g_2 = x_1^2 - x_2$

$g_3 = -x_1 + \frac{1}{2}$

$c^T = (-1, -1)$

How to make sure $\bar{x} = (1, 1)^T$ is optimum for NLP ?

idea: Construct a linear relaxation for (NLP).

i.e. every feasible solution of NLP is feasible for the linear relaxation

Construct linear relaxtion

tight $(\bar{x}) = \{1, 2\}$

$\therefore$ replace $g_1(x) \le 0$ by linear $h_1(x) \le 0$ $(\{x \in \mathbb{R}^2 : g_1(x) \le 0\} \le \{x \in \mathbb{R}^2 : h_1(x) \le 0\}$ $g_1(x) \ge h_1(x))$

$g_2(x) \le 0$ by linear $h_2(x) \le 0$ 同理

drop $g_3(x) \le 0$

1. Prove that $h(x)$ & $g(x)$ touch at $\bar{x}$

Having subgradients $s_1 \in \mathbb{R}^2$ for $g_1(x)$ at $\bar{x}$

$s_2 \in \mathbb{R}^2$ for $g_2(x)$ at $\bar{x}$

we can define $h_1(x) = g_1(\bar{x}) + s_1^T (x - \bar{x})$ ← affine function

$h_2(x) = g_2(\bar{x}) + s_2^T (x - \bar{x})$ ← affine function

Observe that with this definition $h_1(\bar{x}) = g_1(\bar{x}) + s_1^T(\bar{x} - \bar{x}) = g_1(\bar{x})$

$h_2(\bar{x}) = g_2(\bar{x}) + s_2^T(\bar{x} - \bar{x}) = g_2(\bar{x})$

2. Contribute the values

$$\nabla g_1(x) = \begin{bmatrix} -1 \\ 2x_2 \end{bmatrix} \Rightarrow \nabla g_1(\bar{x}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

We compute s_1 as $\nabla g_1(\bar{x})$

$$\Rightarrow h_1(x) = g_1(\bar{x}) + s_1^T(x - \bar{x})$$
$$= 0 + [-1 \ \ 2] \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$
$$= -x_1 + 2x_2 - 1$$

$$\nabla g_2(x) = \begin{bmatrix} 2x_1 \\ -1 \end{bmatrix} \Rightarrow \nabla g_2(\bar{x}) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$\Rightarrow h_2(x) = g_2(\bar{x}) + s_2^T(x - \bar{x})$$
$$= 0 + [2 \ \ -1] \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$
$$= 2x_1 - x_2 - 1$$

Consider the linear relaxtion

$$\min \ -x_1 - x_2 \quad \longleftrightarrow \quad \max \ x_1 + x_2$$

$$\text{s.t.} \quad -x_1 + 2x_2 \le 1$$
$$2x_1 - x_2 \le 1 \qquad \text{(LP)}$$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \bar{x}$ is optimal for (LP) $\Leftrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \text{cone}\left\{\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right\}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ so $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \text{cone}\left\{\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$,

and so $\bar{x}$ is optimal for (LP). so $\bar{x}$ is optimal for (NLP)

- def. subgradient

Let $g: \mathbb{R}^n \to \mathbb{R}$ be a convex function and $\bar{x} \in \mathbb{R}^n$. We say that $s \in \mathbb{R}^n$ is a subgradient of g at point $\bar{x}$ if for every $x \in \mathbb{R}^n$.

$$g(x) \ge g(\bar{x}) + s^T(x - \bar{x}) = \underbrace{g(\bar{x})}_{\text{固定值}} + \underbrace{s^T x}_{\text{linear}} - \underbrace{s^T \bar{x}}_{\text{固定值}}$$

affine.

- proposition

consider a convex (NLP)

$$\boxed{\begin{array}{ll} \min & c^T x \\ \text{s.t} & g_1(x) = 0 \\ & \vdots \\ & g_m(x) \le 0 \end{array}}$$

Let $\bar{x}$ be a feasible solution and let us assume that $g_1 . g_2 \cdots g_m$ are diff'ble at $\bar{x}$, if $-c \in \text{cone}\{\nabla g_i(\bar{x}) : i \in \text{tight}(\bar{x})\}$.
then $\bar{x}$ is optimal for (NLP)

- Slater point

a feasible solution to $\bar{x}$ is a slater point of

$$\boxed{\begin{array}{ll} \min & c^T x \\ \text{s.t} & g_i \le 0 \end{array}}$$

if $g_i(\bar{x}) < 0$. for all $i = 1 \dots k$.

- Theorem (KKT)

consider the convex NLP.

$$\boxed{\begin{array}{ll} \min & c^T x \\ \text{s.t.} & g_1(x) \le 0 \\ & \vdots \\ & g_m(x) \le 0 \end{array}}$$

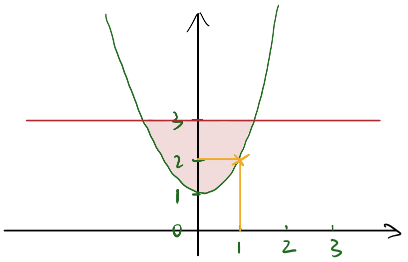
Let $\bar{x}$ be a feasible solution for the NLP. s.t $g_1 \cdots g_m$ are differentiable at $\bar{x}$.

Assume that the NLP has slater point, ie. a point st. $g_1(\tilde{x})<0 \cdots g_m(\tilde{x})<0$

Then $\bar{x}$ is optimal for the NLP $\Leftrightarrow$ $-c \in \text{cone}\{\nabla g_i(\bar{x}) : i \in \text{tight}(\bar{x})\}$

eg. Consider the NLP

$$\boxed{\begin{array}{ll} \min & c^T x \\ \text{s.t.} & g_1(x) = x_1^2 - x_2 + 1 \le 0 \\ & g_2(x) = 2x_1 + x_2 - 4 \le 0 \\ & g_3(x) = x_2 - 3 \le 0 \end{array}}$$



a) determine whether $\bar{x} = (1,2)^T$ is optimal solution when $c = (-3, -1)^T$

tight $(\bar{x}) = \{1, 2\}$.

$\nabla g_1(x) = \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} \rightarrow \nabla g_1(\bar{x}) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

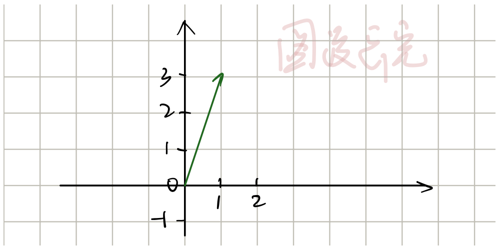
$\nabla g_2(x) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rightarrow \nabla g_2(\bar{x}) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$-c = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \frac{5}{4}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad c = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \in \text{cone}\left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$.

$\therefore \bar{x}$ is optimal $\qquad$ Slatter point not needed for this point.

b) determine whether $\bar{x} = (1,2)^T$ is optimal when $c = (-1, -3)^T$

The NLP has slatter point $\tilde{x} = (0, 2)^T$. So using the computation from part a): $\bar{x}$ is opt $\Leftrightarrow$ $-c = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \in \text{cone}\left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$



Note that $x_1 - x_2 \geq 0$ is satisfied by every point in cone $\left\{\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\}$. but $x_1 - x_2 \geq 0$ is violated by $-c = \begin{pmatrix}1\\3\end{pmatrix}$

So $-c = \begin{pmatrix}1\\3\end{pmatrix} \notin$ cone $\left\{\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\}$.

$\therefore$ c is NOT opt.

Remark , in KKT theorem.

$\bar{x}$ is opt. $\underset{\text{doesn't need slater pt}}{\overset{\text{needs a slater point}}{\rightleftharpoons}}$ $-c \in \text{cone}\{\nabla g_i(\bar{x}) : i \in \text{tight}(\bar{x})\}$.

Recall (last lecture)

For feasible $\bar{x}$ and the NLP. We constructed linear relaxation

$$\boxed{\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & g_1(x) \leq 0 \\ & \quad \vdots \\ & g_m(x) \leq 0 \end{aligned}}$$

$$\boxed{\begin{aligned} \min\ & c^T x \\ \text{s.t}\ & \nabla g_i(\bar{x}) \cdot x \leq 0 \end{aligned}} \quad i \in \text{tight}(\bar{x})$$

KKT tries to say when $\bar{x}$ is opt for LP.

ep. Consider the following NLP where α is a parameter.

$$\boxed{\begin{aligned} \min\ & -x_1 - x_2 \\ \text{s.t}\ & g_1(x) = \alpha x_1 + x_2^2 - 2\alpha - 1 \leq 0 \\ & g_2(x) = 2x_1 + x_2 \leq 7 \\ & g_3(x) = -x_1 + 2x_2^4 \leq 0 \end{aligned}}$$

determine all values of α s.t $\bar{x} = (2\ 1)^T$ is an opt sol.